COMPUTABILITY AND CATEGORICITY OF WEAKLY HOMOGENEOUS BOOLEAN ALGEBRAS AND p-GROUPS

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ABSTRACT. This paper continues the study of weakly homogeneous structures. It is shown that a countable Boolean algebra is weakly homogeneous if and only if it has finitely many atoms. Hence every countable weakly homogeneous Boolean algebra has a computable copy, and a computable Boolean algebra is weakly homogeneous if and only if it is computably categorical. We also characterize countable weakly homogeneous Boolean algebras in various signatures. The countable weakly homogeneous abelian p-groups are characterized, and it is shown that every such group has a computable copy.

1. INTRODUCTION

A structure $\mathscr{A}$ is said to be *ultrahomogeneous*, or simply *homogeneous*, if any isomorphism between finitely generated substructures extends to an automorphism of $\mathscr{A}$. Homogeneous structures were first studied by Fraïssé [7], who defined the *age* of a structure to be the family of finitely generated substructures of $\mathscr{A}$ and gave properties which characterized the age of a homogeneous structure. In [1] we defined the notion of *weakly homogeneous* structures, where $\mathscr{A}$ is weakly homogeneous if there is a finite (*exceptional*) set of elements $a_1,\dots,a_n$ of $\mathscr{A}$ such that any isomorphism between finitely generated substructures $\mathscr{A}$ which maps each a_i to itself may be extended to an automorphism of the structure.

Here are some well-known examples of countable homogeneous structures. See [14, 18] for more details. The linear ordering $(\mathbb{Q},<)$ of the rationals is the unique ultrahomogeneous countable linear ordering. The age here is just the set of all finite linear orderings. An equivalence structure (A,E) is ultrahomogeneous if and only if all equivalence classes have the same size k, $1 \leq k \leq \aleph_0$; then the age is the set of all finite equivalence structures with all classes of size $\leq k$.

This notion turns out to have many connections with computability. For example, Csima, Harizanov, R. Miller and A. Montalban [5] studied computable ages and the computability of the canonical homogeneous structures, called *Fraïssé limits*.

Here we are concerned with countably infinite structures. We say a countable structure (model) $\mathscr{A}$ is computable if its universe A is computable and all of its functions and relations are uniformly computable. Given two computable structures, we will say they are

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computably isomorphic if there exists an isomorphism between them that is computable. For a single computable structure $\mathscr{A}$, we will say $\mathscr{A}$ is computably categorical if every computable structure isomorphic to $\mathscr{A}$ is in fact computably isomorphic to $\mathscr{A}$. More generally, we will say two computable structures are Δ^0_α isomorphic if there exists an isomorphism between them that is Δ^0_α and we will say a computable structure $\mathscr{A}$ is Δ^0_α categorical if every computable structure isomorphic to $\mathscr{A}$ is Δ^0_α isomorphic to $\mathscr{A}$ and will say that a computable structure $\mathscr{A}$ is *relatively* Δ^0_α categorical if, for every structure $\mathscr{B}$ (which is not necessarily computable) isomorphic to $\mathscr{A}$, there is an isomorphism which is Δ^0_α relative to the diagram of $\mathscr{B}$. More generally, an arbitrary structure $\mathscr{A}$ is relatively Δ^0_α categorical if for any structure $\mathscr{B}$ isomorphic to $\mathscr{A}$ there is an isomorphism which is Δ^0_α relative to the diagrams of $\mathscr{A}$ and $\mathscr{B}$. For computable structures, the last two notions agree.

The computability and categoricity of homogeneous structures was explored by the authors in a recent paper [1]. We introduced the notion of weakly ultrahomogeneous (weakly homogeneous) structures and examined several types of countable homogeneous and weakly homogeneous structures. A structure $\mathscr{A}$ is said to be *relatively computably categorical* if,for any structure $\mathscr{B}$ isomorphic to $\mathscr{A}$, there is an isomorphism which is computable from the diagrams of $\mathscr{A}$ and $\mathscr{B}$. Similarly, $\mathscr{A}$ is said to be *relatively* Δ^0_n *categorical* if,for any structure $\mathscr{B}$ isomorphic to $\mathscr{A}$, there is an isomorphism which is Δ^0_n-computable from the diagrams of $\mathscr{A}$ and $\mathscr{B}$. A structure is said to be *locally finite* if every finite set generates a finite substructure (for example, in a pure relational language). Here are the key results from [1], which we will need for the present paper.

Theorem 1.1. *(1) If a structure $\mathscr{A}$ is weakly homogeneous, then it is relatively Δ^0_2 categorical.*

(2) If a structure $\mathscr{A}$ is locally finite and weakly homogeneous, then it is relatively computably categorical.

We then looked at specific classes of structures. Every countable homogeneous linear ordering and also every countable homogeneous equivalence structure has a computable copy. For computable linear orders and computable equivalence structures, the weakly homogeneous structures are exactly the computably categorical structures. For computable injection structures, we observed that there are continuum many homogeneous structures with no computable copy, and there are homogeneous structures which are not computably categorical, although every computably categorical structure is weakly homogeneous. There are continuum many homogeneous trees (T,f) under the predecessor function and they are all are relatively computably categorical. We made a connection between trees under the predecessor function and nested equivalence structures. For these structures, index sets were used to determine the complexity of the property of being homogeneous, and the complexity of the property of being weakly homogeneous.

Characterizations of exceptional sets were given for homogeneous linear orders, equivalence structures, injection structures, and trees as partial orders.

In the present paper, we study weakly homogeneous abelian p-groups and Boolean algebras.

Section 2 contains our results on abelian p-groups. It was shown by Cherlin and Felgner [4] that an abelian p-group is homogeneous if and only if is either divisible or homocylic. Here is the main theorem of this paper.

Theorem 1.2. *An infinite countable abelian p-group $\mathscr{G}$ is weakly homogeneous if and only if it has one of the following forms:*

(1) $\mathscr{G} = \bigoplus_\alpha \mathbb{Z}(p^\infty)$ for some $\alpha \leq \omega$.

(2) $\mathscr{G} = \bigoplus_{i<\omega} \mathbb{Z}(p^n) \oplus \mathscr{F}$*, where n is finite and* $\mathscr{F}$ *is a finite product of finite cyclic groups each having order* $\geq n$*.*

It follows that every countable weakly homogenous abelian p-group has a computable copy. It also follows that every countable weakly homogeneous abelian p-group is computably categorical. Obviously, the converse does not hold.

Section 3 contains our results on Boolean algebras. We consider the standard language $\{\wedge, \vee, \neg, 0, 1\}$ for Boolean algebras as well as subsets of this language. It is well-known that, in the standard language, a Boolean algebra is homogeneous if and only if it is atomless. Here is the second main result.

Theorem 1.3. *A countable Boolean algebra* $\mathscr{B}$ *is weakly homogeneous if and only if it has finitely many atoms.*

It follows that every countable weakly homogeneous Boolean algebra has a computable copy. It also follows that from the results of Goncharov [9] and LaRoche [15], that a computable Boolean algebra is weakly homogeneous if and only if it is computably categorical. If we omit the complement and just use the language $\{\wedge, \vee, 0, 1\}$ of lattices, then $\mathscr{B}$ is homogeneous if and only if it either atomless or has at most 4 elements and $\mathscr{B}$ is weakly homogeneous if and only if it has finitely many atoms. If we view a Boolean algebra as a partial order with the language $\{\leq, 0, 1\}$, then $\mathscr{B}$ is homogeneous if and only if it is finite with at most 4 elements, and $\mathscr{B}$ is weakly homogeneous if and only if it is finite.

2. ABELIAN p-GROUPS

In this section, we will characterize the weakly homogeneous abelian p-groups, and more generally the torsion abelian groups. The computability and categoricity of countable abelian p-groups have been studied for many years.

The computably categorical abelian p-groups were characterized by Goncharov [10] and Smith [22]. We present a slightly stronger formulation of the results as follows. Let us say that $\mathscr{G}$ is *strongly relatively computably categorical* if, for any group $\mathscr{H}$ isomorphic to $\mathscr{G}$, there is an isomorphism which is computable from the diagrams of $\mathscr{G}$ and $\mathscr{H}$.

Theorem 2.1 (Goncharov, Smith)**.** *Let* $\mathscr{G}$ *be an abelian p-group* $\mathscr{G}$ *is strongly relatively computably categorical if and only if either*

(1) $\mathscr{G} \cong \bigoplus_\alpha \mathbb{Z}(p^\infty) \oplus \mathscr{F}$*, where* $\alpha \leq \omega$*, or*
(2) $\mathscr{G} \cong \bigoplus_r \mathbb{Z}(p^\infty) \oplus \bigoplus_\omega \mathbb{Z}(p^m) \oplus \mathscr{F}$*, where* $\mathscr{F}$ *is a finite abelian p-group and* $r, m \in \omega$*.*

Note that the decomposition of $\mathscr{G}$ as a product need not be effective (See Khisamiev [13]). We will refer to $\mathbb{Z}(p^\infty)$ as the quasicyclic p-group, as opposed to the cyclic p-groups $\mathbb{Z}(p^n)$.

For any group of the form

$$\mathscr{G} = \bigoplus_{i<\omega} \mathbb{Z}(p^{n_i}) \oplus \bigoplus_{\alpha} \mathbb{Z}(p^\infty),$$

we define the *character* of $\mathscr{G}$ to be

$$\chi(\mathscr{A}) = \{(n,k) : card(\{i : n_i = n\}) \geq k\}.$$

We say that $\mathscr{G}$ has *bounded character* if for some finite b and all $(n,k) \in \chi(\mathscr{G})$, $n \leq b$, and is said to have *unbounded character* otherwise.

The relatively Δ^0_2 categorical abelian p-groups were characterized by Calvert, Cenzer, Harizanov and Morozov [2] as follows.

Theorem 2.2 (CCHM). *A computable abelian p-group $\mathscr{G} = \bigoplus_{i<\omega} \mathbb{Z}(p^{n_i}) \oplus \bigoplus_\alpha \mathbb{Z}(p^\infty)$ for some $\alpha \leq \omega$ is relatively Δ_2^0 categorical if and only if either*

(1) $\alpha = 0$, that is, every element of $\mathscr{G}$ has finite height, or
(2) $\mathscr{G}$ has bounded character.

The first step in our characterization is to show that every weakly homogeneous abelian p-group is of the form given by Theorem 2.1.

Theorem 2.3. *Let $\mathscr{G}$ be any weakly homogeneous abelian p-group. Then either*

(1) $\mathscr{G} \cong \bigoplus_\alpha \mathbb{Z}(p^\infty) \oplus \mathscr{F}$, where $\alpha \leq \omega$, or
(2) $\mathscr{G} \cong \bigoplus_r \mathbb{Z}(p^\infty) \oplus \bigoplus_\omega \mathbb{Z}(p^m) \oplus \mathscr{F}$, where $\mathscr{F}$ is a finite abelian p-group and $r,m \in \omega$.

Proof. It follows from Theorem 1.1 that every weakly homogeneous abelian p-group $\mathscr{G}$ satisfies the hypothesis of Theorem 2.1 and hence must have one of the forms described above in Theorem 2.1. □

Corollary 2.4. *For any countable weakly homogeneous abelian p-group $\mathscr{G}$, there is a computable group isomorphic to $\mathscr{G}$.*

We will show that in fact the weakly homogeneous abelian p-groups are a proper subset of the relatively computably categorical abelian p-groups.

It is easy to see that a countable homogeneous abelian p-group must either be divisible, that is, of the form $\oplus_\alpha \mathbb{Z}(p^\infty)$ for some $\alpha \leq \omega$, or it must be *homocyclic*, that is, of the form $\bigoplus_\omega \mathbb{Z}(p^n)$, for some fixed n. (See Cherlin and Felgner [4].) Thus every countable homogeneous abelian p-group has a computable copy, and any such copy is computably categorical.

Note in particular that the finite group $\mathbb{Z}(p^m) \oplus \mathbb{Z}(p^n)$ is not homogeneous for any $m \neq n$. To see this, fix $m < n$, and observe that both $(1,0)$ and $(0,p^{n-m})$ have order p^m so there is an isomorphism between the corresponding cyclic subgroups mapping $(1,0)$ to $(0,p^{n-m})$. But this mapping cannot be extended to an automorphism since $(0,p^{n-m})$ is divisible by p but $(1,0)$ is not.

We want to give another way to do this which will prove to be more useful. That is, the elements $a = (0,p)$ and $b = (1,p)$ both have order p^{n-1}, so there is an isomorphism between the corresponding cyclic subgroups $\langle a \rangle$ and $\langle b \rangle$ mapping $(0, jp)$ to (j, jp). This cannot be extended to an automorphism since $(0,p)$ is divisible by p but $(1,p)$ is not. However, it is important to note that it can be extended to an automorphism of $\mathbb{Z}(p^m) \oplus \langle p \rangle$, by mapping $(1,0)$ to $(1,0)$ and hence mapping (i, jp) to $(i+j, jp)$. It can be checked that this mapping is well-defined and one-to-one, using the fact that, for any j, $jp = 0 (mod\, p^n)$ implies $j = 0(mod\, p^{n-1})$ which implies that $j = 0(mod\, p^m)$. Thus for $m < n$, there is a subgroup $\mathscr{A}$ of $\mathbb{Z}(p^m) \oplus \mathbb{Z}(p^n)$ including $\mathbb{Z}(p^m) \oplus 0$ and an isomorphism $f : \mathscr{A} \to \mathscr{A}$ which fixes $\mathbb{Z}(p^m) \oplus 0$ but which cannot be extended to an automorphism. This idea will be used below to show that certain infinite products of cyclic p-groups are not weakly homogeneous.

On the other hand, it is important to note that the roles of $\mathbb{Z}(p^m)$ and $\mathbb{Z}(p^n)$ cannot be reversed here.

We need a bit of terminology. Any element a of a product $\mathscr{G} = \bigoplus_n \mathscr{G}_n$ is an infinite sequence $a = (x_0, x_1, \dots)$ where all but finitely many values x_i are zero. The *support* $Supp(a)$ of a is the finite set $\{n : x_n \neq 0\}$. The support of a finite subset S of $\mathscr{G}$ is $Supp(S) = \cup_{a \in S} Supp(a)$.

Proposition 2.5. *For any $m < n$, $\mathbb{Z}(p^m) \oplus \bigoplus_\omega \mathbb{Z}(p^n)$ is not weakly homogeneous.*

Proof. Let $\mathscr{G} = \mathbb{Z}(p^m) \oplus \bigoplus_\omega \mathbb{Z}(p^n)$. Elements of $\mathscr{G}$ will be infinite sequences $(x_0, x_1, \dots)$ where $x_0 \in \mathbb{Z}(p^m)$ and each x_{i+1} comes from the i^{th} copy of $\mathbb{Z}(p^n)$. Suppose that S is a finite exceptional set. By expanding to a larger (hence still exceptional) set, we may assume that the element $(1, 0, \dots)$ from the $\mathbb{Z}(p^m)$ component is in S and that S is a subgroup of $\mathscr{G}$. Since S is finite, there is some i such that $i+1 \notin Supp(S)$; without loss of generality, we may assume that $1 \notin Supp(S)$. Thus for any $(x_0, x_1, \dots) \in S$, $x_1 = 0$. As in the discussion above, we want to map $a = (0, p, 0, 0, \dots)$ to $b = (1, p, 0, 0, \dots)$, since both have order p^{n-1} and a is divisible by p whereas b is not, so the map would have no extension to an automorphism of $\mathscr{G}$. This map needs to be extended to include S. Let $k = \max Supp(S)$, so that for any $a = (x_0, x_1, \dots) \in S$, and any $i > k$, we have $x_i = 0$. Now define the finite subgroup $\mathscr{A} = \{(x_0, x_1, \dots) \in \mathscr{G} : p \mid x_1 \; \& \; (\forall i > k) x_i = 0\}$ and define the automorphism f of $\mathscr{A}$ by $f(i, jp, x_2, x_3, \dots)$ to $(i+j, jp, x_2, x_3, \dots)$. Since any element $a = (x_0, x_1, \dots)$ of S has $x_1 = 0$, it follows that $f(a) = a$ for any element of S, as desired. Since S was arbitrary, it follows that $\mathscr{G}$ is not weakly homogeneous. □

Next we want to consider products of the form $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^m)$. This is more difficult because we cannot factor out components as we could in the cases above. The following lemma refines the mappings used in the previous arguments, and will be needed.

Lemma 2.6. *For any prime p and any m, n, r with $n = 2m + r + 1$, there is an automorphism ϕ of a subgroup $\mathscr{A}$ of $\mathscr{G} = \mathbb{Z}(p^m) \oplus \mathbb{Z}(p^n)$ which fixes every element of order $\leq p^{m+r}$ (hence fixes all elements of $\mathbb{Z}(p^m) \oplus 0$ and also $0 \oplus \langle p^{m+1} \rangle$), but which cannot be extended to an automorphism*

Proof. Fix p, m, n, r as above and let $\mathscr{A} = \mathbb{Z}(p^m) \oplus p\mathbb{Z}(p^n)$. Define the isomorphism ϕ from $\mathscr{A}$ to itself, as discussed above, by $\phi(i, jp) = (i + j, jp)$. Now suppose that (i, jp) has order $\leq p^{m+r}$. Then $p^{m+r} \cdot jp = 0 (mod\, p^{2m+r+1})$, so that $j = 0 (mod\, p^m)$ and hence $i + j = i (mod\, p^m)$. This shows that elements of order $\leq p^{m+r}$ are fixed by f. As we have seen, ϕ cannot be extended to an automorphism of $\mathscr{G}$ since it maps the divisible element $(0, p)$ to the element $(1, p)$ which is not divisible. □

This leads to the following somewhat surprising result.

Theorem 2.7. *For any finite m, and any s and t with $1 \leq s, t \leq \omega$, the group $\mathscr{G} = \bigoplus_s \mathbb{Z}(p^\infty) \oplus \bigoplus_t \mathbb{Z}(p^m)$ is not weakly homogeneous.*

Proof. Suppose by way of contradiction that $\bigoplus_s \mathbb{Z}(p^\infty) \oplus \bigoplus_t \mathbb{Z}(p^m)$ is weakly homogeneous and let S be an exceptional set. Choose r large enough so that every element of S has order $\leq p^{m+r}$ and let $n = 2m + r + 1$. Now let a be an element of any $\mathbb{Z}(p^\infty)$ component with order p^n, and let b be an element of any $\mathbb{Z}(p^m)$ component with order p^m. The subgroup $\mathscr{B} = \langle a \rangle \oplus \langle b \rangle$ is clearly isomorphic to $\mathbb{Z}(p^n) \oplus \mathbb{Z}(p^m)$. It follows from Lemma 2.6 that there is an automorphism ϕ of $\mathscr{A} = \mathbb{Z}(p^m) \oplus \langle pa \rangle$ which cannot be extended to an automorphism of $\mathscr{B}$. Then ϕ also cannot be extended to an automorphism of $\mathscr{G}$. The simple argument here is that ϕ maps an element $(0, pa)$ which is divisible by p to an element $(1, pa)$ which is not divisible by p. We also observe that S is contained in $\mathscr{B}$ as all elements of S have order $\leq p^{m+r} < p^n$. Thus, ϕ must fix all elements of S. □

This can be extended using the following lemma.

Lemma 2.8. *If the abelian p-group $\mathscr{G} = \mathscr{G}_0 \oplus \mathscr{G}_1$ is weakly homogeneous, and $\mathscr{G}_1$ is finite, then $\mathscr{G}_0$ is weakly homogeneous.*

Proof. Let $C = \{(a_1,b_1),\dots,(a_n,b_n)\}$ be an exceptional set for $\mathscr{G}$; we may assume that $0\oplus\mathscr{G}_1$ is a subgroup of C and that C is a subgroup of $\mathscr{G}$. Let $A = \{a_1,\dots,a_n\}$, which will be a subgroup of $\mathscr{G}_0$. Now suppose that ϕ_0 is an isomorphism from a subgroup $\mathscr{F}_0$ of $\mathscr{G}_0$ to a subgroup $\mathscr{H}_0$ of $\mathscr{G}_0$, where $A \subseteq \mathscr{F}_0 \cap \mathscr{H}_0$ and $\phi_0(a_i) = a_i$ for $i = 1,\dots,n$. We can extend this to an isomorphism ϕ from $\mathscr{F}_0 \oplus \mathscr{G}_1$ to $\mathscr{H}_0 \oplus \mathscr{G}_1$ by letting $\phi(x,y) = (\phi_0(x),y)$. Then $\phi(a_i,b_i) = (a_i,b_i)$ for $i = 1,\dots,n$. Since C is an exceptional set for $\mathscr{G}$, it follows that there is an automorphism ψ of $\mathscr{G}_0\oplus\mathscr{G}_1$ extending ϕ. Then the mapping $\psi_0 : \mathscr{G}_0 \to \mathscr{G}_0$ defined by $\psi_0(x) = \psi(x,0)$ is an automorphism of $\mathscr{G}_0$ extending ϕ_0. □

Corollary 2.9. *For any nonzero finite group $\mathscr{F}$ and any α with $0 < \alpha \le \omega$, $\bigoplus_\alpha \mathbb{Z}(p^\infty)\oplus\mathscr{F}$ is not weakly homogeneous.*

To complete our characterization of the weakly homogeneous abelian p-groups, we need the following extension lemma.

Lemma 2.10. *Suppose that $\mathscr{A},\mathscr{B},\mathscr{C}$ are abelian p-groups, that $\mathscr{B} \subseteq \mathscr{C}$ and that f is an automorphism of $\mathscr{A}\oplus\mathscr{B}$ with $f(0,b) = (0,b)$ for all $b \in \mathscr{B}$. Then the map f may be extended to an automorphism g of $\mathscr{A}\oplus\mathscr{C}$, such that $g(0,c) = (0,c)$ for all $c \in \mathscr{C}$.*

Proof. We just let $g(a,c) = f(a,0)+(0,c)$. This is clearly a homomorphism from $\mathscr{A}\oplus\mathscr{C}$ to itself. It agrees with f on $\mathscr{A}\oplus\mathscr{B}$, since $g(a,b) = f(a,0)+(0,b) = f(a,0)+ f(0,b) = f(a,b)$. To see that g is one-to-one, suppose that $g(a,c) = (0,0)$ and let $f(a,0) = (a_1,b_1)$. Then $(a_1,b_1 +c) = (0,0)$. It follows that $a_1 = 0$ and $b_1 +c = 0$. But this means that $c \in \mathscr{B}$ and therefore $(0,0) = g(a,c) = f(a,0)+(0,c) = f(a,c)$. Since f is one-to-one, it follows that $a = c = 0$. To check that g is onto, let $(a_1,c_1) \in \mathscr{A}\oplus\mathscr{C}$. Since f is onto, there exist $a \in \mathscr{A}$ and $b \in \mathscr{B}$ such that $f(a,b) = (a_1,0)$. Then $g(a,b + c_1) = g(a,b) + g(0,c_1) = f(a,b)+g(0,c_1) = (a_1,0)+(0,c_1) = (a_1,c_1)$. □

Theorem 2.11. *For any n and any finite product $\mathscr{K}$ of cyclic p-groups each having order $> p^n$, $\mathscr{G} = \bigoplus_{i<\omega}\mathbb{Z}(p^n)\oplus\mathscr{K}$ is weakly homogeneous.*

Proof. We let $S = 0\oplus\mathscr{K}$ be our exceptional set. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be finite subgroups of $\mathscr{G}$ each including S and let $h : \mathscr{G}_1 \to \mathscr{G}_2$ be an isomorphism with $h(0,k) = (0,k)$ for all $k \in \mathscr{K}$. Our goal is to show that h can be extended to an automorphism F of $\mathscr{G}$ which fixes S.

We can write $\mathscr{G}_i = \mathscr{A}_i \oplus\mathscr{K}$, where $\mathscr{A}_i = \{a \in \bigoplus_{i<\omega}\mathbb{Z}(p^n) : (a,0) \in \mathscr{G}_i\}$, for $i = 1,2$. Let $\mathscr{K} = \mathbb{Z}(p^{n_1})\oplus\cdots\oplus\mathbb{Z}(p^{n_k})$, where each $n_j > n$. Consider the subgroup $\mathscr{K}^- = p^{n_1-n}\mathbb{Z}(p^{n_1})\oplus p^{n_2-n}\mathbb{Z}(p^{n_2})\oplus\cdots\oplus p^{n_k-n}\mathbb{Z}(p^{n_k})$ of $\mathscr{K}$. $\mathscr{K}^-$ is isomorphic to k copies of $\mathbb{Z}(p^n)$ and it consists exactly of the elements of $\mathscr{K}$ of order $\le p^n$. Thus $\mathscr{A}_1\oplus\mathscr{K}^-$ is the set of elements of $\mathscr{G}_1$ of order $\le p^n$ and must be mapped by h to $\mathscr{A}_2\oplus\mathscr{K}^-$, the elements of $\mathscr{A}_2\oplus\mathscr{K}$ of order $\le p^n$. Now $\mathscr{G}^- = \bigoplus_{i<\omega}\mathbb{Z}(p^n)\oplus\mathscr{K}^-$ is isomorphic to $\bigoplus_{i<\omega}\mathbb{Z}(p^n)$ and is therefore homogeneous. It follows that there is an automorphism f of $\mathscr{G}^-$ which extends the restriction of h to $\mathscr{A}_1\oplus\mathscr{K}^-$, and thus $f(0,k) = (0,k)$ for all $k \in \mathscr{K}^-$. Finally, by Lemma 2.10, f may be extended to an automorphism of $\mathscr{G}$, say F, such that $F(0,k) = (0,k)$ for all $k \in \mathscr{K}$. In that case, it is easy to see that F extends h. □

We are now ready to state the characterization of weakly homogeneous abelian p-groups:

Theorem 2.12. *An infinite countable abelian p-group $\mathscr{G}$ is weakly homogeneous if and only if it has one of the following forms:*

(1) $\mathscr{G} = \bigoplus_\alpha\mathbb{Z}(p^\infty)$ *for some* $\alpha \le \omega$.
(2) $\mathscr{G} = \bigoplus_{i<\omega}\mathbb{Z}(p^n)\oplus\mathscr{F}$, *where n is finite and $\mathscr{F}$ is a finite product of cyclic p-groups each having order $> p^n$.*

In the second case, an exceptional set may be given to contain a generator for each factor of $\mathscr{F}$.

Proof. Suppose first that $\mathscr{G}$ is weakly homogeneous. Then it has one of the two forms specified by Theorem 2.3. If $\mathscr{G}$ has the first such form, then by Corollary 3.3 it must have form (1). If $\mathscr{G}$ has the second form from Theorem 2.3, then by Theorem 2.7 and Lemma 2.8 it must have no quasicyclic component, and then it follows from Proposition 2.5 and Lemma 2.8 that $\mathscr{G}$ must have form (2).

Next suppose that $\mathscr{G}$ has the prescribed form. If $\mathscr{G}$ is of form (1), then $\mathscr{G}$ is in fact homogeneous. If $\mathscr{G}$ is of form (2), then $\mathscr{G}$ is weakly homogeneous by Theorem 2.11. □

On the other hand, it is clear from Theorem 2.1 of Goncharov and Smith that not every computably categorical abelian p-group is weakly homogeneous.

We point out a similar, and similarly named, concept which nevertheless differs from our definition of weakly homogeneous. Melnikov and Ng [17] defined an abelian p-group to have the weak homogeneity property (WHP) if it is either divisible, or for each non-zero element a of order p and finite height, there exist at most finitely many elements with order p and height greater than that of a. They prove that this is equivalent to an abelian p-group being of the form $U \oplus H$ where U is any finite direct sum of cyclic and quasi-cyclic p-groups, H is a direct power of some fixed cyclic or quasi-cyclic p-group $\mathbb{Z}(p^{\lambda})$, and the least α such that $\mathbb{Z}(p^{\alpha})$ occurs in U (if there are any) is at least λ. So every abelian p-group with the WHP is (relatively) computably categorical. It follows from Theorem 2.12 that any weakly homogeneous p-group has the weak homogeneity property. On the other hand, the group $\mathbb{Z}(p^{\infty}) \oplus \mathbb{Z}(p)$ has the WHP but is not weakly homogeneous. Melnikov and Ng use the WHP property to show that the notion of computable categoricity for computable torsion abelian groups is Π^0_4 complete.

2.1. **Torsion abelian groups.** Finally, we want to briefly consider arbitrary torsion abelian groups. Any such group $\mathscr{G}$ may be represented as a direct sum $\bigoplus_p \mathscr{G}_p$ where $\mathscr{G}_p$ is the subgroup of $\mathscr{G}$ of elements having order p^n for some n.

Theorem 2.13. *For any torsion abelian group $\mathscr{G}$:*

1. *(1) $\mathscr{G}$ is homogeneous if and only if $\mathscr{G}_p$ is homogeneous for each prime p;*
2. *(2) $\mathscr{G}$ is weakly homogeneous if and only if $\mathscr{G}_p$ is weakly homogeneous for each prime p and $\mathscr{G}_p$ is homogeneous for all but finitely many primes.*

Proof. (1) Assume that $\mathscr{G}$ is homogeneous. Fix p and let ϕ be an isomorphism between two finitely generated subgroups $\mathscr{A}$ and $\mathscr{B}$ of $\mathscr{G}_p$. Then $\mathscr{A}$ and $\mathscr{B}$ are also subgroups of $\mathscr{G}$ and therefore ϕ may be extended to an automorphism ψ of $\mathscr{G}$. Then the restriction of ψ to $\mathscr{G}_p$ is clearly an automorphism of $\mathscr{G}_p$ which extends ϕ.

Next assume that each $\mathscr{G}_p$ is homogeneous. Suppose that $\mathscr{A}$ and $\mathscr{B}$ are finitely generated subgroups of $\mathscr{G}$ and that $\phi : \mathscr{A} \to \mathscr{B}$ is an isomorphism. Then $\mathscr{A}$ and $\mathscr{B}$ are finite and there is some finite set P of primes such that $\mathscr{A} = \bigoplus_{p \in P} \mathscr{A}_p$ and $\mathscr{B} = \bigoplus_{p \in P} \mathscr{B}_p$. It follows that ϕ induces isomorphisms $\phi_p : \mathscr{A}_p \to \mathscr{B}_p$ for each $p \in P$. Now, for each $p \in P$, $\mathscr{A}_p$ and $\mathscr{B}_p$ are finitely generated subgroups of G_p. Since each $\mathscr{G}_p$ is homogeneous, it follows that ϕ_p may be extended to an automorphism ψ_p of $\mathscr{G}_p$. Now we can define an automorphism ψ of $\mathscr{G}$ extending ϕ by letting ψ be the identity on each $\mathscr{G}_p$ for $p \notin P$.

(2) Assume that $\mathscr{G}$ is weakly homogeneous with exceptional finite subgroup $\mathscr{H} = \bigoplus_p \mathscr{H}_p$. Fix p and let ϕ be an isomorphism between two finitely generated subgroups $\mathscr{A}$ and $\mathscr{B}$ of $\mathscr{G}_p$ which fixes each element of $\mathscr{H}_p$. Now extend ϕ to an isomorphism of $\mathscr{A} \oplus \bigoplus_{q \neq p} \mathscr{H}_q \mapsto \mathscr{B} \oplus \bigoplus_{q \neq p} \mathscr{H}_q$ by fixing each element of $\mathscr{H}_q$ for all $q \neq p$. This may

now be extended to an automorphism ψ of $\mathscr{G}$ and the restriction of ψ to $\mathscr{G}_p$ will be an automorphism which extends ϕ and fixes $\mathscr{H}_p$. For all but finitely many primes p, $\mathscr{H}_p = \emptyset$, and it follows that for those p, $\mathscr{G}_p$ is homogeneous.

For the other direction, suppose that for some finite set P of primes, $\mathscr{G}_p$ is weakly homogeneous with exceptional subgroup $\mathscr{H}_p$ and that $\mathscr{G}_p$ is homogeneous for all other primes p. It is easy to see that $\mathscr{G}$ will be weakly homogeneous with exceptional subgroup $\bigoplus_{p\in P} \mathscr{H}_p$. □

The condition that $\mathscr{G}_p$ is homogeneous for almost every component p cannot be dropped in part (2) of Theorem 2.13, as the following example shows.

Example. Let $\mathscr{G} = \bigoplus_p \left(\mathbb{Z}(p) \oplus \mathbb{Z}(p^2)\right)$. Then $\mathscr{G}_p = \left(\mathbb{Z}(p) \oplus \mathbb{Z}(p^2)\right)$ is a finite group and is weakly homogeneous but not homogeneous for every p. We check that $\mathscr{G}$ is not weakly homogeneous. Let $\mathscr{H}$ be any potential finite exceptional subgroup of $\mathscr{G}$. Then for some finite set P of primes, $\mathscr{H} \subseteq \bigoplus_{p\in P} \left(\mathbb{Z}(p) \oplus \mathbb{Z}(p^2)\right)$. Now let q be any prime not in P and consider the subgroups $\mathscr{A} = \mathscr{H} \oplus \mathbb{Z}(q) \oplus 0$ and $\mathscr{B} = \mathscr{H} \oplus 0 \oplus q\mathbb{Z}(q^2)$ of $\bigoplus_{p\in P\cup\{q\}} \left(\mathbb{Z}(p) \oplus \mathbb{Z}(p^2)\right)$ and consider the isomorphism from $\mathscr{A} \mapsto \mathscr{B}$ fixing each element of $\mathscr{H}$ and mapping $(x,0)$ to $(0,xq)$ for the components corresponding to $\mathbb{Z}(q) \oplus \mathbb{Z}(q^2)$. This cannot be extended to an automorphism of $\mathscr{G}$ since $(0,q)$ is divisible by q but $(1,0)$ is not.

3. Boolean Algebras

The standard signature for the study of Boolean algebras is $\{\wedge, \vee, \neg, 0, 1\}$. Here $a \wedge b$ is the *meet* of elements a and b, $a \vee b$ is the *join*, $\neg a$ is the *complement* of a, and 0 and 1 are the identity elements. A Boolean algebra $\mathscr{B} = (B, \wedge, \vee, \neg, 0, 1)$ may be viewed as a 2-group under the operation $a \oplus b = (a \wedge \neg b) \vee (b \wedge \neg a)$. $\mathscr{B}$ also has a natural partial ordering, defined by $a \leq b \iff a \vee b = b$.

The computability-theoretic properties of Boolean algebras have been well studied. The computably categorical Boolean algebras were determined independently by LaRoche [15] and Goncharov [9].

Theorem 3.1. *(LaRoche, Goncharov) A computable Boolean algebra is computably categorical iff it is relatively computably categorical iff it has finitely many atoms.*

Further investigations on the existence of effective isomorphisms appear in [20] and [16]. It is well known that a countable Boolean algebra is homogeneous if it is atomless or has size at most 4. Details can be found in [11]. So we proceed to the weakly homogeneous case.

Theorem 3.2. *Assume the language is $\{\wedge, \vee, \neg, 0, 1\}$. Then a countable Boolean algebra $\mathscr{B}$ is weakly homogeneous if and only if $\mathscr{B}$ has finitely many atoms.*

Proof. Suppose first that $\mathscr{B}$ is weakly homogeneous. It follows from Theorem 1.1 that $\mathscr{B}$ is relatively computably categorical. Then by Theorem 3.1, $\mathscr{B}$ has finitely many atoms.

Suppose next that the Boolean algebra $\mathscr{C}$ has only finitely many atoms. We can clearly take $\mathscr{C}$ to be infinite (otherwise it is trivially weakly homogeneous). Then we may assume that $\mathscr{C}$ has the form $\mathscr{B} \otimes \mathscr{A}$ where $\mathscr{B}$ is atomless and $\mathscr{A}$ is finite. Let $\{0\} \otimes \mathscr{A}$ be the exceptional set. Now suppose that we are given an isomorphism $F : \mathscr{C}_1 \to \mathscr{C}_2$ where $\mathscr{C}_1$ and $\mathscr{C}_2$ are finite subalgebras of $\mathscr{C}$ which both include $\{0\} \otimes \mathscr{A}$ and such that $F(0,a) = (0,a)$ for all $a \in \mathscr{A}$. Let $\mathscr{B}_1 = \pi_0(\mathscr{C}_1)$ and $\mathscr{B}_2 = \pi_0(\mathscr{C}_2)$, where $\pi_0(x,y) = x$. Note that $\mathscr{C}_1 = \mathscr{B}_1 \otimes \mathscr{A}$ and $\mathscr{C}_2 = \mathscr{B}_2 \otimes \mathscr{A}$ and that (b,a) is an atom of $\mathscr{C}_i$ if and only if either $b = 0$ and a is an atom of $\mathscr{A}$ or $a = 0$ and b is an atom of $\mathscr{B}_i$.

Now define $G(x) = \pi_0(F(x,0))$ for $x \in \mathscr{B}_1$. G is a composition of homomorphisms and therefore is a homomorphism. To see that G is one-to-one, suppose that $G(x) = G(y)$ for some $x,y \in \mathscr{B}_1$, and thus $F(x,0) = (b,i)$ and $F(y,0) = (b,j)$ for some $i,j \in \mathscr{A}$. Then $F(x \oplus y, 0) = (0, i \oplus j) = F(0, i \oplus j)$, so that $x \oplus y = 0$ and hence $x = y$ and also $i = j$. This shows that G is one-to-one.

We claim that, in addition, $F(x,0) = (G(x),0)$ for all $x \in \mathscr{B}_1$, and thus $F(x,i) = (G(x),i)$ for all x,i. In particular, since F is onto, this claim will show that $G : \mathscr{B}_1 \to \mathscr{B}_2$ is also onto. Since $\mathscr{B}_1$ is finite, it suffices to show this for all atoms b of $\mathscr{B}_1$. Let $F(b,0) = (y,i)$. Since $(b,0)$ is an atom of $\mathscr{C}_1$, (y,i) must be an atom of $\mathscr{C}_2$. If $i \neq 0$, then y must equal 0, otherwise (y,i) cannot be an atom of $\mathscr{C}_2$. But then $F(b,0) = (0,i) = F(0,i)$, so that $b = i = 0$. Therefore, $i = 0$ and hence $F(b,0) = (y,0) = (G(b),0)$.

Since $\mathscr{B}$ is homogeneous, G may be extended to an automorphism of $\mathscr{B}$. Finally, let $H : \mathscr{C} \to \mathscr{C}$ be given by $H(x,i) = (G(x),i))$. This is an automorphism of $\mathscr{C}$ which extends F. □

Corollary 3.3. *A computable Boolean algebra $\mathscr{B}$ is weakly homogeneous iff $\mathscr{B}$ is computably categorical iff $\mathscr{B}$ is relatively computably categorical.*

Corollary 3.4. *Every countable weakly homogeneous Boolean algebra has a computable copy.*

The previous proof shows that a finite subalgebra of a weakly homogeneous Boolean algebra is exceptional when it includes every atom. It is natural to conjecture that including all the atoms is not only sufficient, but also necessary. However, the following proposition show this is not the case. This suggests a classification of the exceptional sets for weakly homogeneous Boolean algebras would require a finer combinatorial analysis like for linear orders and trees as found in [1].

Proposition 3.5. *Suppose that $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$, where both $\mathscr{A}$ and $\mathscr{B}$ are homogeneous. (That is, each is either atomless are has size ≤ 4.) Then $S = \{(0,1)\}$ is an exceptional set for $\mathscr{C}$.*

Proof. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be finite subalgebras of $\mathscr{C}$ each containing $(0,1)$, and hence also containing the complement $(1,0)$. Let $F : \mathscr{C}_1 \to \mathscr{C}_2$ be an isomorphism with $F(0,1) = (0,1)$ and $F(1,0) = (1,0)$. Let $\mathscr{A}_i$ be the projection $\pi_0(\mathscr{C}_i)$ of $\mathscr{C}_i$ onto $\mathscr{A}$ and let $\mathscr{B}_i = \pi_1(\mathscr{C}_i)$, for $i = 1,2$.

Claim 3.6. *(1) For any $a \in \mathscr{A}$, any $b \in \mathscr{B}$, and for $i = 1,2$, $a \in \mathscr{A}_i \iff (a,0) \in \mathscr{C}_i$, and similarly $b \in \mathscr{B}_i \iff (0,b) \in \mathscr{C}_i$.*
(2) $\mathscr{C}_i = \mathscr{A}_i \otimes \mathscr{B}_i$ for $i = 1,2$.

Proof. For part (1), certainly $(a,0) \in \mathscr{C}_i$ implies that $a \in \mathscr{A}_i$. Now suppose that $a \in \mathscr{A}_i$. Then for some $b \in B$, $(a,b) \in \mathscr{C}_i$. But then $(a,0) = (a,b) \wedge (1,0) \in \mathscr{C}_i$ as well, since $(1,0) \in \mathscr{C}_i$. Similarly, $b \in \mathscr{B}_i$ implies that some $(a,b) \in \mathscr{C}_i$ and hence $(0,b) = (a,b) \wedge (0,1) \in \mathscr{C}_i$.

For part (2), it is clear that $\mathscr{C}_i \subseteq \mathscr{A}_i \otimes \mathscr{B}_i$. Now suppose that $(x,y) \in \mathscr{A}_i \otimes \mathscr{B}_i$. Then by part (1), $(x,0) \in \mathscr{C}_i$ and $(0,y) \in \mathscr{C}_i$. Thus $(x,y) = (x,0) \vee (0,y) \in \mathscr{C}_i$. □

Define $G : \mathscr{A}_1 \to \mathscr{A}_2$ by $G(a) = \pi_0(F(a,0))$ and define $H : \mathscr{B}_1 \to \mathscr{B}_2$ by $H(b) = \pi_1(F(0,b))$.

Claim 3.7. *(3) For any $(a,b) \in \mathscr{C}$, $F(a,b) = (G(a),H(b))$.*
(4) G and H are isomorphisms.

Proof. For part (3), let $F(a,b) = (c,d)$. Since $F(1,0) = (1,0)$ and $(a,0) = (a,b) \wedge (1,0)$, it follows that $F(a,0) = (c,d) \wedge (1,0) = (c,0)$, so that $c = G(a)$. Similarly, $d = H(b)$.

Next we prove part (4). To see that G is a homomorphism, let $G(a) = b$ and $G(c) = d$. Then, by part (3), $F(a,0) = (b,0)$ and $F(c,0) = (d,0)$. Thus $F(a \vee c, 0) = (b \vee d, 0)$ and hence $G(a \vee c) = b \vee d$. Similarly $F(a \wedge c, 0) = (b \wedge d, 0)$ so $G(a \wedge c) = b \wedge d$. A similar argument shows that H is a homomorphism. To see that G is one-to-one, suppose that $G(a) = G(c)$. Then $F(a,0) = F(c,0)$, so that $a = c$. In the same way we can see that H is one-to-one. To see that G is onto, let $c \in \mathscr{A}_2$. Then $(c,0) \in \mathscr{C}_2$, so that $F(a,b) = (c,0)$ for some $(a,b) \in \mathscr{C}_1$. It follows from part (3) that $c = G(a)$. Similarly H is onto. □

Since $\mathscr{A}$ and $\mathscr{B}$ are homogeneous, it follows that there are automorphisms $\hat{G} : \mathscr{A} \to \mathscr{A}$ extending G and $\hat{H} : \mathscr{B} \to \mathscr{B}$ extending H. Thus for $a \in \mathscr{A}_1$ and $b \in \mathscr{B}_1$, we have $\hat{G}(a) = G(a)$ and $\hat{H}(b) = H(b)$. Now let $\hat{F}(a,b) = (\hat{G}(a), \hat{H}(b))$. Then $\hat{H}$ is an automorphism of $\mathscr{C}$ and furthermore, for $(a,b) \in \mathscr{C}_1$, we have $\hat{F}(a,b) = (G(a), H(b)) = F(a,b)$. □

It should be mentioned that there is a natural correspondence between Boolean algebras and linear orders. Given a (computable) linear order $\mathscr{L}$, the left-closed, right-open intervals of $\mathscr{L}$ generate a (computable) Boolean algebra called the interval algebra of $\mathscr{L}$. That is, for each $a < b$ in $\mathscr{L}$, let $[a,b) = \{x : a \le x < b\}$ and let $\mathscr{B}(\mathscr{L})$ be the family of finite unions of such intervals and their complements. It is easy to see that a linear order has finitely many successivities iff its interval algebra has finitely many atoms. So from [1] it immediately follows that a countable linear order is weakly homogeneous iff its interval algebra is weakly homogeneous. However, it does not seem possible to use this correspondence to directly obtain a proof of Theorem 3.2. . More discussion about the interval algebra of a linear ordering in the context of computability can be found in [6] and [19].

Next we consider other signatures for Boolean algebras.

By omitting the complement operation, Boolean algebras with the meet and join operations may be viewed as bounded, distributive lattices in the signature $\{\vee, \wedge, 0, 1\}$. Note that the complement $\neg a$ is definable in the lattice as the unique element x such that $x \vee a = 1$ and $x \wedge a = 0$.

Boolean algebras come with a natural ordering definable from the algebraic operations, where $a \le b$ iff $a \wedge b = a$ iff $a \vee b = b$, so may be considered as partial orderings in the signature $\{\le, 0, 1\}$. Note that the join and meet operations are definable in the partial ordering. For example, $a \vee b = c$ means that $a \le c$ and $b \le c$ and that for any d such that $a \le d$ and $b \le d$, $c \le d$.

We will classify the homogeneous and weakly homogeneous Boolean algebras as lattices and as partial orderings.

Proposition 3.8. *(1) If $\mathscr{B}$ is homogeneous as a lattice or as a partial order, then it is also homogeneous as a Boolean algebra.*

(2) If $\mathscr{B}$ is weakly homogeneous as a lattice or as a partial order, then it is also weakly homogeneous as a Boolean algebra.

Proof. Suppose that $\mathscr{B}$ is homogeneous in one of the two other signatures and let $\mathscr{B}_1$ and $\mathscr{B}_2$ be finitely generated subalgebras and $F : \mathscr{B}_1 \to \mathscr{B}_2$ an isomorphism. Then $\mathscr{B}_1$ and $\mathscr{B}_2$ are in fact finite, and hence are finitely generated in the smaller signature as well. Hence F may be extended to an automorphism H of $\mathscr{B}$ as a partial order, or as a lattice. But this means that H is in fact an automorphism of $\mathscr{B}$ as a Boolean algebra. Next suppose that $\mathscr{B}$ is weakly homogeneous in a smaller signature with some exceptional set S and suppose that $F : \mathscr{B}_1 \to \mathscr{B}_2$ is an isomorphism of finitely generated substructures each including S

and fixing every element of S. Then F is also an isomorphism in the smaller signature, and hence may be extended to an automorphism H of $\mathscr{B}$ in the smaller signature. As above, H is in fact an automorphism of $\mathscr{B}$ as a Boolean algebra. □

We can now classify the homogeneous and weakly homogeneous Boolean algebras as lattices. We will need a fact about the lattices and Boolean algebras they generate.

Lemma 3.9. *([12] Section II.4.1) Let $\mathscr{B}$ be a Boolean algebra in the language $\{\wedge, \vee, 0, 1\}$ and let $\varphi : L_1 \to L_2$ be an isomorphism of sublattices. Then φ extends to an isomorphism of subalgebras of $\mathscr{B}$ generated by L_1, L_2.*

Proposition 3.10. *For any countable Boolean algebra $\mathscr{B}$, $\mathscr{B}$ is homogeneous in the language of Boolean algebras if and only if it is homogeneous in the language of lattices, and similarly for weakly homogeneous.*

Proof. In each case, one direction is immediate from Proposition 3.8. Now suppose that $\mathscr{B}$ is homogeneous as a Boolean algebra and let $\phi : L_1 \to L_2$ be an isomorphism of two finitely generated sublattices. By Lemma 3.9, ϕ extends to an isomorphism F of the subalgebras $\mathscr{B}_1$ and $\mathscr{B}_2$ generated by L_1 and L_2. Since $\mathscr{B}$ is homogeneous as a Boolean algebra, F extends to an automorphism H of $\mathscr{B}$ as a Boolean algebra. Then H is also a lattice automorphism and extends ϕ. If L_1 and L_2 are required to include some exceptional set S and ϕ is required to have $\phi(x) = x$ for $x \in S$, then $\mathscr{B}_1$ and $\mathscr{B}_2$ above will also include S and the extension F above will still have $F(x) = x$ for $x \in S$. So F will have again have an extension to an automorphism H of $\mathscr{B}$ as a Boolean algebra. □

We next consider Boolean algebras in the language of partial orders $\{\leq, 0, 1\}$; the classification here might be a bit surprising.

Proposition 3.11. *Assume the language is the language of partial orders $\{\leq, 0, 1\}$. Then given a countable Boolean algebra $\mathscr{B}$,*

(1) $\mathscr{B}$ is homogeneous if and only if $\mathscr{B}$ is finite with at most 4 elements.
(2) $\mathscr{B}$ is weakly homogeneous if and only if $\mathscr{B}$ is finite.

Proof. (1): If $\mathscr{B}$ has $1, 2$ or 4 elements then it is clearly homogeneous. Now suppose $\mathscr{B}$ has at least 8 elements. Then $\mathscr{B}$ contains elements a and b such that $0_{\mathscr{B}} < a < b < 1_{\mathscr{B}}$. Let $c = a \vee \neg b$. We now claim that $c \not\geq b$. Suppose not, then $c \wedge (b \wedge \neg a) = b \wedge \neg a$. Since $c \wedge (b \wedge \neg a) = 0_{\mathscr{B}}$ this means that $b \wedge \neg a = 0_{\mathscr{B}}$ and hence $a \geq b$, a contradiction. So b and c are incomparable. Now we can consider $\phi(b) = b$ and $\phi(c) = \neg b$ which is an isomorphism of the suborderings $\{0_{\mathscr{B}}, 1_{\mathscr{B}}, b, c\}$ and $\{0_{\mathscr{B}}, 1_{\mathscr{B}}, b, \neg b\}$. This cannot be extended to an automorphism of $\mathscr{B}$ as a partial order because a is a non-zero element below both b and c, while no such element exists below b and $\neg b$.

(2): Every finite set is trivially weakly homogeneous. Now assume that $\mathscr{B}$ is an infinite Boolean algebra. Fix a finite subalgebra $\mathscr{A}$ of $\mathscr{B}$. We can find an infinite set C so that for any $x, y \in C$ and any $a \in \mathscr{A}$, $x < a \iff y < a$ and $a < x \iff a < y$, since there are only finitely many such configurations. Let m be the meet of all elements of $\mathscr{A}$ above every element of C and let j be the join of all elements of $\mathscr{A}$ below C. Then all elements of C come from the interval $[j, m]$. Using elements of C we can generate from C, relativized to $[j, m]$, a subalgebra of $\mathscr{B}$ with 8 elements; here the complement $\neg x = m \wedge \neg x$. So this algebra has m as its largest element and j as its smallest element. Now repeat the argument from the homogeneous case. This means we obtain elements $x, y, z \in C$ so x is incomparable with both y and z, but while x, y have an element below them in $\mathscr{B}$ and strictly above j, the pair x, z does not. Then the partial isomorphism fixing all of $\mathscr{A}$, fixing x, and sending y to z

can not extend to an automorphism of $\mathscr{B}$. So $\mathscr{A}$ is not exceptional, hence $\mathscr{B}$ is not weakly homogeneous. □

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