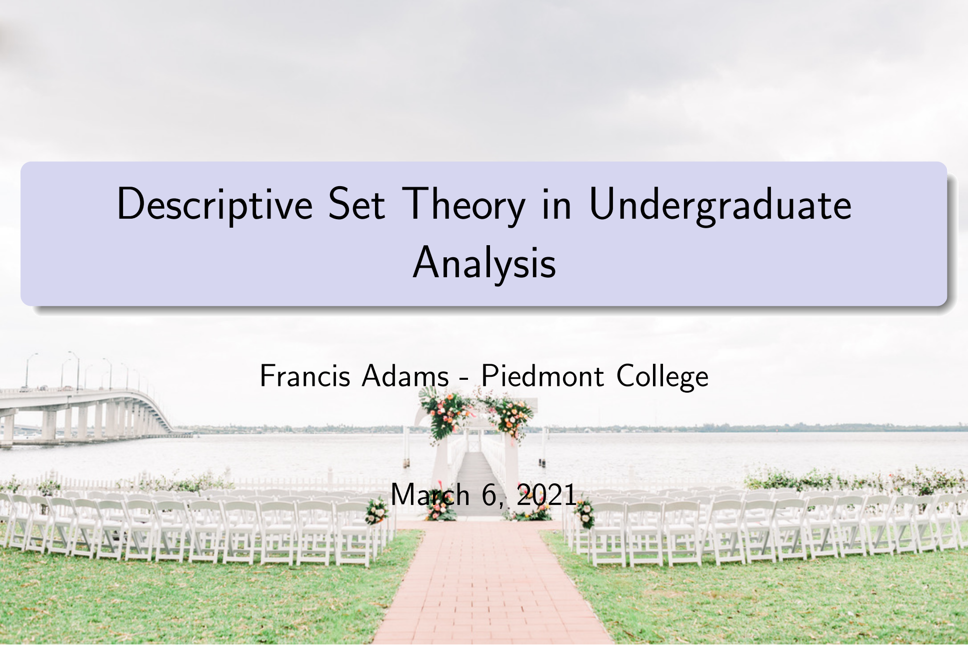

Descriptive Set Theory in Undergraduate Analysis

Francis Adams - Piedmont College

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Plan for the Talk

In this talk I will discuss how descriptive set theory can

1. Provide additional examples to be used in class or for student projects.
2. Unify with a common theme several results that already commonly appear in undergraduate analysis.
3. Serve as a stand-alone second course in analysis that introduces ideas from topology and foundations.

Descriptive Set Theory

What is descriptive set theory?

The study of sets and functions from analysis and topology concerning their complexity and definability using techniques from set theory and mathematical logic.

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By 'complexity and definability' we will mean how the set or function is constructed from or defined in terms of simpler objects.

Open Sets

Recall that $O \subseteq \mathbb{R}$ is open if for all $x \in O$ there is $\epsilon > 0$ so $B_\epsilon(x) = \{y : |x - y| < \epsilon\} \subseteq O$.

Every nonempty open set is

- a unique disjoint union of open intervals.
- a union of intervals with rational endpoints.
- uncountable, with the same cardinality as $\mathbb{R}$.

Closed Sets and Perfect Sets

What can we say about the structure of closed sets?

Definition

A set is closed if its complement is open. A set $P \subseteq \mathbb{R}$ is perfect if it is closed and has no isolated points, i.e. every point is a limit point.

Examples of closed sets are $[a, b]$, $\mathbb{Z}$, $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, 0\}$, the Cantor Middle-Thirds Set.

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As a corollary, the Continuum Hypothesis holds for closed sets: every closed set is either countable or has the cardinality of $\mathbb{R}$.

More Complex Sets of Reals

Open sets are closed under arbitrary unions and finite intersections, while closed sets are closed under finite unions and arbitrary intersections.

Definition

Say a set is F_σ if it is a countable union of closed sets, and say a set is G_δ if it is the countable intersection of open sets.

Open or closed sets are both F_σ and G_δ.
All countable sets are F_σ.
For any function $f : \mathbb{R} \to \mathbb{R}$, the set of points where f is continuous is G_δ.

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But what if you take a countable union of G_δ sets or a countable intersection of F_σ sets?

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Definition

Let the Borel sets be the smallest collection of subsets of $\mathbb{R}$ which

- *Includes the open sets.*
- *Is closed under countable unions and intersections.*
- *Is closed under complements.*

We can think of these as starting on the level of open/closed sets, and taking unions, intersections, and complements to get more complicated sets.

Borel Sets

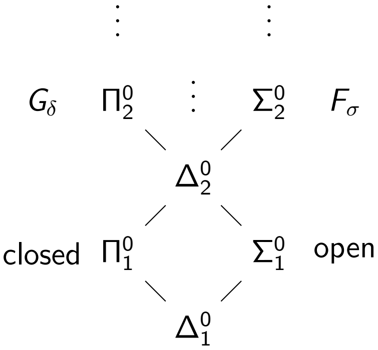
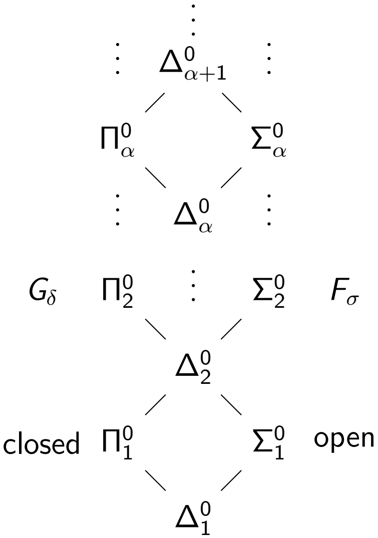


Figure: Borel Hierarchy

Borel Sets


$\Delta^0_{\alpha+1}$
Π^0_α
Σ^0_α
Δ^0_α
G_δ
Π^0_2
Σ^0_2
F_σ
Δ^0_2
closed
Π^0_1
Σ^0_1
open
Δ^0_1

Borel Functions

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous iff for every open set $O \subseteq \mathbb{R}$, $f^{-1}(O)$ is open.

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A function $f : \mathbb{R} \to \mathbb{R}$ is Borel iff for every open set $O \subseteq \mathbb{R}$, $f^{-1}(O)$ is Borel.

So we can measure the complexity of a Borel function by the complexity of its preimages $f^{-1}(O)$.

Question: How do we 'build up' Borel functions in the way we 'built up' Borel sets?

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Another Question: The pointwise limit of continuous functions need not be continuous. What functions *can* be obtained by starting with continuous functions and repeatedly taking pointwise limits?

Answer:

- $f : \mathbb{R} \to \mathbb{R}$ is a pointwise limit of continuous functions iff $f^{-1}(O)$ is F_σ for all open sets O. (Such functions are called Baire class 1 functions.)
- If $(f_n : n \geq 0)$ are Baire class 1 functions and converge uniformly to f, then f is Baire class 1.

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- $f : \mathbb{R} \to \mathbb{R}$ is a pointwise limit of continuous functions iff $f^{-1}(O)$ is F_σ for all open sets O. (Such functions are called Baire class 1 functions.)
- If $(f_n : n \geq 0)$ are Baire class 1 functions and converge uniformly to f, then f is Baire class 1.
- The set of Borel functions on $\mathbb{R}$ is the smallest set of functions which includes the continuous functions and is closed under taking pointwise limits.

Continuity of Borel Functions

For any function, the set of points where it is continuous is G_δ.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is Borel, then f is continuous on a dense G_δ set.

Derivatives

Question: Derivatives need not be continuous, but do satisfy the Intermediate Value Theorem. How discontinuous can they be?

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Answer:

If f is differentiable, f' is Borel (in fact Baire class 1). So f' is continuous on a dense G_δ set.

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4. Baire Category Theorem, Meager Sets, Baire Measurable Sets.

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3. Universality Results: e.g. every Polish space is a continuous image of $\mathbb{N}^{\mathbb{N}}$.
4. Baire Category Theorem, Meager Sets, Baire Measurable Sets.
5. Impossibility Results: e.g. The Borel hierarchy has uncountable length, there is a Baire Measurable but non-Borel set, there is a non-Baire Measurable set.

Thank you!

- Understanding Analysis, Stephen Abbott, Undergraduate Texts in Mathematics
- Classical Descriptive Set Theory, Alexander Kechris, Graduate Texts in Mathematics
- Stanford Encyclopedia of Philosophy: The Early Development of Set Theory (Critical Period) https://plato.stanford.edu/entries/settheory-early/

There are open sets that aren't closed and closed sets that aren't open. What about for F_σ and G_δ?

Theorem (Baire Category Theorem)

The intersection of countably many open dense sets is dense (so nonempty). Equivalently the union of countably many closed nowhere dense sets is nowhere dense (so not $\mathbb{R}$).

As a corollary, $\mathbb{Q}$ is not G_δ and the set of irrationals is not F_σ.

Say a set is

- meager if it is contained in the union of countably many closed nowhere dense sets.
- non meager if it is not meager.
- comeager if its complement is meager, equivalently if it contains a dense G_δ set.
- Baire measurable if it can be represented as $O \triangle M$ where O is open and M is meager.