

Construction of Directed Strongly Regular Graphs Using Block Matrices

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July 31, 2009

Abstract

The concept of directed strongly regular graphs was introduced by Duval in “A Directed Graph Version of Strongly Regular Graphs” [*Journal of Combinatorial Theory*, Series A 47 (1988) 71 – 100]. Duval also provided several construction methods for directed strongly regular graphs. We construct several new classes of directed strongly regular graphs with parameters $\lambda = \mu = t - 1$ or $\lambda + 1 = \mu = t$. The directed strongly regular graphs reported in this paper are obtained using a block construction of adjacency matrices of regular tournaments and circulant matrices. We then give some algebraic and combinatorial interpretation of these graphs in connection with known directed strongly regular graphs and related combinatorial structures.

Keywords Directed Strongly Regular Graphs, Cayley Graphs, Regular Tournaments, Doubly-Regular Team Tournaments

AMS Classification

1 Introduction

This paper investigates directed strongly regular graphs and some methods of constructing them from block matrices. Section 2 introduces the necessary notation and defines directed strongly regular graphs in terms of its adjacency matrix and its combinatorial properties. Section 3 looks at feasibility conditions of parameter sets established by Duval and some construction methods he provides. We also describe a DSRG construction method not used by Duval, construction using Cayley Graphs of groups. Some of these Cayley Graph construction motivate the first constructions in Section 4 based on regular tournaments. After these and other constructions stemming from block matrices, especially regular tournaments, Section 5 investigates when these constructions produce isomorphic graphs, including using different tournaments in one construction and the same tournament in different constructions. Section 7 is a summarizing list of all our construction methods and the parameter sets satisfied by each, as well as a short list of the first few parameter sets for which we have found new constructions.

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2 Preliminaries

All graphs considered in this paper will be finite simple graphs; so our graphs will have no loops or multiple edges. Let Γ be a directed graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For any $x, y \in V(\Gamma)$, we say that x is adjacent to y , denoted $x \rightarrow y$, if there is an edge from x to y . There may also be an edge from y to x , in which case we will say there is an undirected edge between x and y , written as $x \leftrightarrow y$. Finally, x is not adjacent to y , signified as $x \nrightarrow y$, if neither $x \rightarrow y$ nor $x \leftrightarrow y$.

Let Γ be a graph with $V(\Gamma) = \{x_1, x_2, \dots, x_n\}$. The adjacency matrix $A = A(\Gamma)$ of Γ is an $n \times n$ matrix whose rows and columns are indexed by the vertices such that

$$A_{ij} = \begin{cases} 1 & \text{if } x_i \text{ is adjacent to } x_j \\ 0 & \text{otherwise} \end{cases}$$

We will use Γ and its adjacency matrix A interchangeably.

A strongly regular graph with parameters (n, k, λ, μ) is an undirected graph with n vertices whose adjacency matrix A satisfies the following equations:

$$A^2 = kI + \lambda A + \mu(J - I - A)$$

$$AJ = JA = kJ$$

where I is the identity matrix and J is the all-ones matrix. From the first equation, we see that the number of paths of length two from a vertex x to another vertex y is λ if x and y are adjacent, μ if x and y are not adjacent. This second equation means that each vertex has valency k .

The concept of ‘strong regularity’ in the class of directed graphs is a generalization of that in the class of undirected graphs. Let Γ be a directed graph with its adjacency matrix A . The graph Γ is called a *directed strongly regular graph with parameters* (n, k, t, λ, μ) , denoted $\text{DSRG}(n, k, t, \lambda, \mu)$, if A satisfies the following equations:

$$A^2 = tI + \lambda A + \mu(J - I - A)$$

$$AJ = JA = kJ$$

Thus, each vertex of $\text{DSRG}(n, k, t, \lambda, \mu)$ has k out-neighbors and k in-neighbors, including t neighbors counted as both in- and out-neighbors of the vertex. For vertices $x \neq y$, there are λ paths of length two from x to y if $x \rightarrow y$ and μ paths of length two if $x \nrightarrow y$.

A strongly regular graph with parameters (n, k, λ, μ) is viewed as a $\text{DSRG}(n, k, t, \lambda, \mu)$, characterized by a DSRG with $t = k$. A DSRG with $t = 0$ is a graph known as doubly-regular tournament. An example of a DSRG that is not a strongly regular graph has the parameters $(8, 3, 2, 1, 1)$ with adjacency matrix A below. This DSRG is illustrated in Figure 1.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

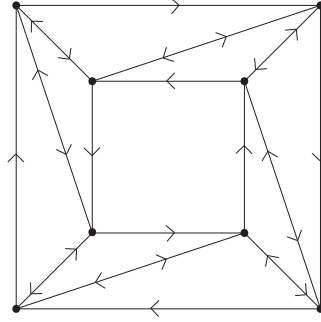


Figure 1

3 Feasibility Conditions and Known Construction Methods

It is immediate from the definition of directed strongly regular graphs that $0 \leq \mu, \lambda$ and $0 \leq t \leq k \leq n-1$. However, we will only consider the ('genuine') DSRGs with $0 < t < k$, excluding totally undirected ($t = k$) and totally directed ($t = 0$) cases. We are able to put further restrictions on parameter sets, which determines if it is feasible for a DSRG with certain parameters to exist.

It is known that if Γ is a DSRG(n, k, t, λ, μ) with adjacency matrix A , then the complement Γ' of Γ is a DSRG($n, k', t', \lambda', \mu'$) with adjacency matrix $A' = J - I - A$ where

$$k' = (n - 2k) + (k - 1)$$

$$\lambda' = (n - 2k) + (\mu - 2)$$

$$t' = (n - 2k) + (t - 1)$$

$$\mu' = (n - 2k) + \lambda.$$

This can easily be shown by evaluating A'^2 .

Duval showed that if a DSRG(n, k, t, λ, μ) is not totally undirected or complete, nor totally directed, then the following equations and inequalities hold true:

$$k(k + (\mu - \lambda)) = t + (n - 1)\mu$$

$$(\mu - \lambda)^2 + 4(t - \mu) = d^2$$

$$d \mid 2k - (\mu - \lambda)(n - 1)$$

$$\frac{2k - (\mu - \lambda)(n - 1)}{d} \equiv n - 1 \pmod{2}$$

$$\left| \frac{2k - (\mu - \lambda)(n - 1)}{d} \right| \leq n - 1$$

$$0 \leq \lambda < t < k$$

$$0 < \mu \leq t < k$$

$$-2(k - t - 1) \leq \mu - \lambda \leq 2(k - t)$$

These parameter restrictions allow for a list of the feasible directed strongly regular graphs to be compiled. This list immediately suggests to the observer that no DSRG of prime order exists, which is in fact the case (cf. [2]).

Duval provided an initial list of feasible parameter sets in his paper [2], but a more complete list is available in [1].

Duval described many different construction methods including three that we will describe here:

1. Constructing directed strongly regular graphs using quadratic residue
2. Constructing DSRGs with a block construction using permutation matrices
3. Constructing DSRGs by using the Kronecker product to construct new DSRGs from smaller ones

The first construction uses quadratic residue matrices to construct $\text{DSRG}(n, k, t, \lambda, \mu)$ with parameters $(2q, q-1, \frac{1}{2}(q-1), \frac{1}{2}(q-1)-1, \frac{1}{2}(q-1))$, where $q = 4m+1$ and is a prime power. The adjacency matrices of such DSRGs will take the form

$$A = \begin{bmatrix} Q & C_1 \\ C_2 & Q \end{bmatrix}.$$

C_1 and C_2 are σ_1 and σ_2 circulant matrices respectively, where a σ circulant matrix C satisfies $C_{ij} = C_{i-k, j-\sigma k}$. This means that each row, or each column, is equal to the previous row (column) shifted σ entries to the right (down). Q is a quadratic residue matrix of order q , indexed by the elements of $\text{GF}(q)$, the Galois Field of order q . When R is the set of quadratic residues of $\text{GF}(q)$, the nonzero elements $x \in \text{GF}(q)$ such that $x = y^2$ for some $y \in \text{GF}(q)$, and N is the set of quadratic non-residues of $\text{GF}(q)$, all other nonzero elements of $\text{GF}(q)$, Q is defined by

$$Q_{ij} = \begin{cases} 1 & \text{if } i - j \in R \\ 0 & \text{if } i - j \in N \end{cases}.$$

This construction method produces a DSRG iff

- $\sigma_1\sigma_2 = 1 \in \text{GF}(q)$
- $\sigma_1, \sigma_2 \in N$
- The partition of $\text{GF}(q)^*$ into the two sets, each of $2m$ elements,

$$S = \{x \in \text{GF}(q)^* : (C_2)_{0,x} = 1\} \quad \text{and} \quad T = \{x \in \text{GF}(q)^* : (C_2)_{0,x} = 0\}$$

described by the first row satisfies the following ‘‘difference partition’’ property: Each of the $4m$ elements of $\text{GF}(q)^*$ occurs exactly m times in the $4m^2$ differences $s - t$ where $s \in S$ and $t \in T$.

An example of an adjacency matrix for the $\text{DSRG}(10, 4, 2, 1, 2)$ using the preceding construction is

$$A = \left[\begin{array}{ccccc|ccccc} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right].$$

DSRG’s with parameter $(2(2\mu+1), 2\mu, \mu, \mu-1, \mu)$ can be found by another, simpler, block construction using matrices of the form

$$A = \begin{bmatrix} Q & PQ \\ (PQ)^T & Q \end{bmatrix}$$

where

$$Q + Q^T = J - I$$

$$QJ = JQ = \mu J$$

and P is a permutation matrix with rank 2, so

$$PJ = JP = J$$

$$P = P^T = P^{-1}.$$

This construction method yields a DSRG iff $PQ = (PQ)^T$.

An example of an adjacency matrix for the DSRG(14, 6, 3, 2, 3) using the preceding construction is

$$A = \left[\begin{array}{cccccc|cccccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array} \right].$$

A final construction developed by Duval uses the Kronecker product of matrices. A small example of how the Kronecker product, $A \otimes B$, works is shown below. If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

their Kronecker Product

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}.$$

The construction method works as follows: let A be the adjacency matrix of a DSRG and J_m be the all-ones matrix. For $m > 1$, $A \otimes J_m$ is the adjacency matrix of a DSRG($nm, km, tm, \lambda m, \mu m$) iff $t = \mu$. a DSRG with the same parameters as above can also be constructed from $J_m \otimes A$.

We will discuss some other construction methods not introduced by Duval. We begin with an introduction to constructing DSRGs using Cayley graphs. Let G be a finite group and $S \subset G - \{e\}$. The Cayley Graph of G generated by S , $Cay(G; S)$, is the digraph Γ such that $V(\Gamma) = G$ and

$$E(\Gamma) = \{(x, y) : \exists s \in S \text{ such that } xs = y\}.$$

An example of a Cayley graph that is from an abelian group is illustrated in Figure 2. An example of a DSRG constructed from a Cayley graph is illustrated in Figure 3.

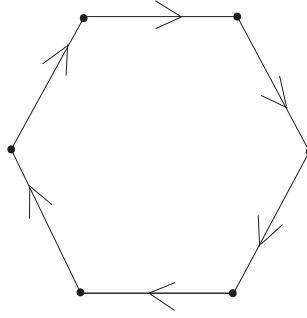


Figure 2, $\text{Cay}(\mathbb{Z}_6, \{1\})$

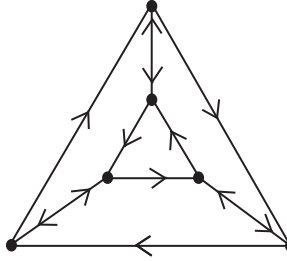


Figure 3, $\text{Cay}(S_3, \{(12), (123)\})$

Cayley graphs of groups give us another way to construct DSRGs. It is clear that if $\text{Cay}(G; S)$ is to be a $\text{DSRG}(n, k, t, \lambda, \mu)$, $|G| = n$ and $|S| = k$. Also, it is necessary that in the induced multiplication table of S :

- The identity of the group, e , appears t times. An easier way to check this is, if $S^{-1} = \{x \in G : x^{-1} \in S\}$, then $|S \cap S^{-1}| = t$
- Each element of S appears λ times.
- Each of the elements of $G - S - \{e\}$ appears μ times.

A result by Jørgensen shows that if G is abelian, then $\text{Cay}(G; S)$ is not a DSRG for any $S \subset G$ [6].

Other constructions of DSRGs as Cayley graphs were developed by Hobart and Shaw [5]. They used the dihedral group $D_{2n} = \langle \alpha, \beta : \beta^2 = \alpha^n = e \text{ and } \beta\alpha\beta = \alpha^{-1} \rangle$. They showed how $\text{Cay}(D_{2n}, S)$ can be a DSRG as follows:

1. When $n = 2\lambda$, an even integer, they constructed the DSRG $(4\lambda, 2\lambda - 1, \lambda, \lambda - 1, \lambda - 1)$ by setting $S = \{\alpha, \alpha^2, \dots, \alpha^{\lambda-1}, \beta, \beta\alpha, \dots, \beta\alpha^{\lambda-1}\}$.
2. For $n = 2\lambda + 1$, an odd integer, they constructed a DSRG $(4\lambda + 2, 2\lambda + 1, \lambda, \lambda - 1, \lambda)$ by letting $S = \{\alpha, \alpha^2, \dots, \alpha^\lambda, \beta, \beta\alpha, \dots, \beta\alpha^\lambda\}$.

These graphs also appear in [8]. Analyzing the adjacency matrices of these graphs, we observe that these adjacency matrices can be expressed as block matrices of the form

$$B = \begin{bmatrix} A & A^T \\ A & A^T \end{bmatrix}$$

where A is the adjacency matrix of a highly structured graph, namely, a regular tournament. We observe that this form of block matrix is indeed able to be used in a general construction method for DSRGs.

4 New Constructions using Tournaments and Circulant Matrices

In this section we will introduce several new methods of constructing directed strongly regular graphs that fall into three categories.

1. Constructing DSRGs using regular tournaments
2. Constructing DSRGs using doubly regular tournaments
3. Constructing DSRGs using circulant matrices

Definition 4.1. *A tournament is a directed graph Γ such that for any $x, y \in V(\Gamma)$ exactly one of $x \rightarrow y$ or $y \rightarrow x$ holds. A tournament Γ is said to be regular if every vertex in $V(\Gamma)$ has the same out-degree. Thus a regular tournament has $n = 2k + 1$ if n and k denote the number of vertices and the valency of the graph, respectively.*

The adjacency matrix A of a tournament Γ satisfies the equation $A + A^T = J - I$. If Γ is a regular tournament with valency k , then $JA = AJ = kJ$.

Lemma 4.2. *If A is an adjacency matrix of a regular tournament with valency k , then*

1. $B = \begin{bmatrix} A & A^T \\ A & A^T \end{bmatrix}$
2. $C = \begin{bmatrix} A & A \\ A^T & A^T \end{bmatrix}$

are adjacency matrices of directed strongly regular graphs with parameters $(4k + 2, 2k, k, k - 1, k)$.

Proof. Let J denote the $(4k + 2) \times (4k + 2)$ all-ones matrix while \bar{J} denote the $(2k + 1) \times (2k + 1)$ all-ones matrix, and similarly for I and \bar{I} . Then we have

$$JB = BJ = \begin{bmatrix} A & A^T \\ A & A^T \end{bmatrix} \begin{bmatrix} \bar{J} & \bar{J} \\ \bar{J} & \bar{J} \end{bmatrix} = 2kJ.$$

Since $A\bar{J} = \bar{J}A = k\bar{J}$ and $A + A^T + \bar{I} = \bar{J}$,

$$B^2 + B = \begin{bmatrix} A^2 + A^T A + A & AA^T + (A^T)^2 + A^T \\ A^2 + A^T A + A & AA^T + (A^T)^2 + A^T \end{bmatrix} = kJ.$$

Equivalently,

$$B^2 = kI + (k - 1)B + k(J - I - B).$$

Therefore B is an adjacency matrix of a DSRG $(4k + 2, 2k, k, k - 1, k)$. Similarly, it can be shown that matrix C is also the adjacency matrix of a DSRG $(4k + 2, 2k, k, k - 1, k)$. \square

A similar block construction also produces DSRGs and is actually closely related to the previous construction.

Lemma 4.3. *If the matrix A is the adjacency matrix of a regular tournament of order $2k + 1$, then the matrix*

$$M(A) = \begin{bmatrix} A & A^T + I \\ A + I & A^T \end{bmatrix}$$

is the adjacency matrix of a DSRG with parameters $(4k + 2, 2k + 1, k + 1, k, k + 1)$.

Proof. From Lemma 4.2, we know that the adjacency matrix $B = \begin{bmatrix} A & A^T \\ A & A^T \end{bmatrix}$ is the adjacency matrix of a DSRG with parameters $(4k + 2, 2k, k, k - 1, k)$. Duval showed that the complement of a DSRG is also a DSRG (see Section 3). We will show that graph Γ' represented by M is the complement of the graph Γ represented by B .

Since A is an adjacency matrix of a regular tournament, it holds $A + A^T = J - I$. Therefore, the complement B' of B can be simplified to

$$\begin{bmatrix} J - I - A & J - A^T \\ J - A & J - I - A^T \end{bmatrix} = \begin{bmatrix} A^T & A + I \\ A^T + I & A \end{bmatrix}.$$

If we choose P to be the permutation matrix equal to $\begin{bmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{bmatrix}$ where $\mathbf{0}$ denotes the all-zeros matrix.

Then

$$PB'P = \begin{bmatrix} A & A^T + I \\ A + I & A^T \end{bmatrix} = M.$$

Therefore, $B' \cong M$ and M is the adjacency matrix of DSRG with parameters $(4k+2, 2k+1, k+1, k, k+1)$. A completely analogous construction coming from Lemma 4.2.2 also produces DSRGs in a nice block matrix form. □

To simplify our next few constructions, we will use the notation $M(A)$ to mean the matrix of the form $\begin{bmatrix} A & A^T + I \\ A + I & A^T \end{bmatrix}$ for any matrix A .

This construction method can be generalized by having multiple columns or rows of A and A^T for a regular tournament.

Lemma 4.4. *If A is an adjacency matrix of a regular tournament with valency k , then*

$$1. B = \begin{bmatrix} A & A^T & A & \dots & A^T \\ A & A^T & A & \dots & A^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A & A^T & A & \dots & A^T \end{bmatrix}$$

$$2. C = \begin{bmatrix} A & A & \dots & A \\ A^T & A^T & \dots & A^T \\ A & A & \dots & A \\ \vdots & \vdots & \ddots & \vdots \\ A^T & A^T & \dots & A^T \end{bmatrix}$$

are adjacency matrices of DSRGs with parameters $((4k + 2)w, 2kw, kw, (k - 1)w, kw)$ where w is the number of A and A^T blocks in each column (or row).

Proof. Since $B = \begin{bmatrix} A & A^T \\ A & A^T \end{bmatrix}$ is the adjacency matrix of a DSRG $(4k + 2, 2k, k, k - 1, k)$, using Duval's Kronecker product construction, $J_w \otimes B$ is also the adjacency matrix of a DSRG, but with parameters $((4k + 2)w, 2kw, kw, (k - 1)w, kw)$. The proof that C is a DSRG $(4k + 2, 2k, k, k - 1, k)$ is exactly the same. □

Definition 4.5. A regular tournament T is said to be doubly regular if for every vertex $x \in V(T)$, the out-neighbors of x span a regular tournament. If T is a doubly regular tournament of order n , with regular valency k and the degree of the induced subgraph on the out-neighbors λ , then $n = 2k + 1 = 4\lambda + 3$.

Definition 4.6. An (m, r) -team tournament is a digraph obtained from the complement $\overline{m \circ K_r}$ of m copies of the complete graph K_r by giving an orientation in such a way that every undirected edge $\{x, y\}$ is assigned with either $x \rightarrow y$ or $x \leftarrow y$ but not both.

We note that an (m, r) -team tournament has m maximal independent sets of size r , and the edges are directed links between the vertices of distinct maximal independent sets.

Definition 4.7. An (m, r) -team tournament Γ with adjacency matrix A is said to be doubly regular iff

- (1) every vertex of Γ has in-degree and out-degree $k = \frac{1}{2}(m - 1)r$, and
- (2) there exist positive integers α, β and γ such that for every pair of distinct vertices x and y , the number of directed paths of length 2 from x to y is

$$\begin{cases} \alpha & \text{if } x \rightarrow y \\ \beta & \text{if } x \leftarrow y \\ \gamma & \text{otherwise} \end{cases} .$$

As in [7], the adjacency matrix of a doubly regular (m, r) -team tournament satisfies the following equations

- (1) $AJ = JA = kJ$;
- (2) $A^2 = \alpha A + \beta A^T + \gamma(J - I - A - A^T)$.

In [7], we can find $(m, 2)$ -team tournaments coming from doubly regular tournaments of order $m - 1$. Let A be an adjacency matrix of a doubly regular tournament T of order $m - 1 = 2k + 1 = 4\lambda + 3$. Then

$$D(T) = \left[\begin{array}{c|ccc|c|ccc} 0 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ \hline 0 & & & & 1 & & & \\ \vdots & & A & & \vdots & & & A^T \\ \hline 0 & & & & 1 & & & \\ \hline 0 & 0 & \dots & 0 & 0 & 1 & \dots & 1 \\ \hline 1 & & & & 0 & & & \\ \vdots & & A^T & & \vdots & & & A \\ \hline 1 & & & & 0 & & & \end{array} \right]$$

is an adjacency matrix of a doubly regular $(m, 2)$ -team tournament.

Lemma 4.8. Let $D = D(T)$ be an $(m, 2)$ -team tournament described above with $m = 2k + 2 = 4\lambda + 4$, then

$$M = M(D) = \begin{bmatrix} D & D^T + I \\ D + I & D^T \end{bmatrix}$$

is an adjacency matrix of a DSRG with parameters

$$(4m, 2m - 1, m, m - 1, m - 1) = (16\lambda + 16, 8\lambda + 7, 4\lambda + 4, 4\lambda + 3, 4\lambda + 3).$$

Proof. Being an adjacency matrix of doubly regular $(m, 2)$ -team tournament, it is shown in [7] that D satisfies

$$(a) D^2 = (2\lambda + 1)(D + D^T) + (4\lambda + 3)(J - I - D - D^T),$$

$$(b) DD^T = (4\lambda + 3)I + (2\lambda + 1)(D + D^T).$$

First we will show that $MJ = JM = kJ = (2m - 1)J$

$$MJ = \begin{bmatrix} D\bar{J} + D^T\bar{J} + \bar{J} & D\bar{J} + D^T\bar{J} + \bar{J} \\ D\bar{J} + D^T\bar{J} + \bar{J} & D\bar{J} + D^T\bar{J} + \bar{J} \end{bmatrix} = JM$$

Using the definition of a regular tournament we can see that since the doubly regular tournament D has valency $m - 1$

$$MJ = JM = \begin{bmatrix} \bar{J}(m - 1 + m - 1 + 1) & \bar{J}(m - 1 + m - 1 + 1) \\ \bar{J}(m - 1 + m - 1 + 1) & \bar{J}(m - 1 + m - 1 + 1) \end{bmatrix} = (2m - 1)J$$

Therefore the equation $MJ = JM = kJ$ holds.

We now square the adjacency matrix M to show that M is an adjacency matrix of the desired DSRG.

$$M^2 = \left[\begin{array}{c|c} D^2 + D^T D + D + D^T + I & DD^T + D + D^T + D^{T^2} \\ \hline D^2 + D^T D + D + D^T & DD^T + D + D^T + D^{T^2} + I \end{array} \right].$$

Simplifying this by using the equalities (a) and (b) as well as $D + D^T = J - I$ yields

$$M^2 = \left[\begin{array}{c|c} (4\lambda + 3)J + I & (4\lambda + 3)J \\ \hline (4\lambda + 3)J & (4\lambda + 3)J + I \end{array} \right].$$

Therefore, $t = 4\lambda + 4$, $\lambda = \mu = 4\lambda + 3$. Each vertex will have in- and out-valency equal to $2m - 1$ because each doubly regular $(m, 2)$ -team tournament has valency $m - 1$. Each DSRG will have $4m$ vertices because in a doubly regular $(m, 2)$ -team tournament the number of vertices equals $2m$ and in the adjacency matrix M , thus $v = 4m$. Therefore, we have shown that we can use a doubly regular $(m, 2)$ team tournament to construct a DSRG with the parameters $(4m, 2m - 1, m, m - 1, m - 1)$ where $m \equiv 0 \pmod{4}$. \square

We can extend this construction in lemma 4.8 to make use of any regular tournament instead of only doubly regular tournaments.

Lemma 4.9. *Let A be the adjacency matrix of a regular tournament T of order h and*

$$D = D(T) = \begin{bmatrix} 0 & \mathbf{1}^T & 0 & \mathbf{0}^T \\ \mathbf{0} & A & \mathbf{1} & A^T \\ 0 & \mathbf{0}^T & 0 & \mathbf{1}^T \\ \mathbf{1} & A^T & \mathbf{0} & A \end{bmatrix}$$

where $\mathbf{0}$ and $\mathbf{1}$ denote the n -dimensional column vectors of all zeros and all ones, respectively. The matrix

$$M(D) = \begin{bmatrix} D & D^T + I \\ D + I & D^T \end{bmatrix}$$

is the adjacency matrix of a DSRG $(4(h + 1), 2h + 1, h + 1, h, h)$ where $h \equiv 1 \pmod{2}$.

Proof. First we will show that $MJ = JM = kJ = (2h + 1)J$

$$MJ = \begin{bmatrix} D\bar{J} + D^T\bar{J} + \bar{J} & D\bar{J} + D^T\bar{J} + \bar{J} \\ D\bar{J} + D^T\bar{J} + \bar{J} & D\bar{J} + D^T\bar{J} + \bar{J} \end{bmatrix} = JM$$

Using the definition of a regular tournament we can see that since the regular tournament D has valency m then

$$MJ = JM = \begin{bmatrix} \bar{J}(h+h+1) & \bar{J}(h+h+1) \\ \bar{J}(h+h+1) & \bar{J}(h+h+1) \end{bmatrix} = (2h+1)J$$

Therefore the equation $MJ = JM = kJ$ holds.

Since $A + A^T = J_h - I_h$, we have

$$D + D^T = \left[\begin{array}{cc|cc} 0 & \mathbf{1}^T & 0 & \mathbf{1}^T \\ \mathbf{1} & A + A^T & \mathbf{1} & A + A^T \\ \hline 0 & \mathbf{1}^T & 0 & \mathbf{1}^T \\ \mathbf{1} & A + A^T & \mathbf{1} & A + A^T \end{array} \right] = \left[\begin{array}{cc|cc} J_{h+1} - I_{h+1} & J_{h+1} - I_{h+1} \\ \hline J_{h+1} - I_{h+1} & J_{h+1} - I_{h+1} \end{array} \right].$$

$$D^T D = DD^T = \left[\begin{array}{c|c|c|c} h & k\mathbf{1}^T & 0 & k\mathbf{1}^T \\ \hline k\mathbf{1} & J_h + A^T A + AA^T & k\mathbf{1} & A^2 + A^{T^2} \\ \hline 0 & k\mathbf{1}^T & n & k\mathbf{1} \\ \hline k\mathbf{1} & A^2 + A^{T^2} & k\mathbf{1} & J_n + AA^T + A^T A \end{array} \right]$$

$$D^2 = (D^T)^2 = \left[\begin{array}{c|c|c|c} 0 & k\mathbf{1}^T & h & k\mathbf{1}^T \\ \hline k\mathbf{1} & A^2 + A^{T^2} & k\mathbf{1} & J_h + A^T A + AA^T \\ \hline h & k\mathbf{1}^T & 0 & k\mathbf{1} \\ \hline k\mathbf{1} & J_h + A^T A + AA^T & k\mathbf{1} & A^2 + A^{T^2} \end{array} \right]$$

$$D^2 + DD^T + D + D^T =$$

$$\left[\begin{array}{c|c|c|c} h & (2k+1)\mathbf{1}^T & h & (2k+1)\mathbf{1}^T \\ \hline (2k+1)\mathbf{1} & (A + A^T)^2 + J_h + A + A^T & n\mathbf{1} & (A + A^T)^2 + J_h + A + A^T \\ \hline h & (2k+1)\mathbf{1}^T & h & (2k+1)\mathbf{1} \\ \hline (2k+1)\mathbf{1} & (A + A^T)^2 + J_h + A + A^T & h\mathbf{1} & (A + A^T)^2 + J_h + A + A^T \end{array} \right]$$

Using the fact that for a regular tournament of order h with adjacency matrix A , $A + A^T = J_h - I_h$ we can easily simplify $D^2 + DD^T + D + D^T$ to hJ_{2h+2}

From lemma 4.8, it is known that

$$M(D)^2 = \left[\begin{array}{c|c} \frac{D^2 + D^T D + D + D^T + I}{D^2 + D^T D + D + D^T} & \frac{DD^T + D + D^T + D^{T^2}}{DD^T + D + D^T + D^{T^2} + I} \end{array} \right].$$

Using the above simplification we can transform

$$M^2 = \left[\begin{array}{c|c} \frac{hJ_{2h+2} + I_{2h+2}}{hJ_{2h+2}} & \frac{hJ_{2h+2}}{hJ_{2h+2} + I_{2h+2}} \end{array} \right]$$

into

$$hJ_{4n+4} + I_{4h+4}$$

Therefore M is the adjacency matrix of the DSRG with parameters $(4(h+1), 2h+1, h+1, h, h) = (8(k+1), 4k+3, 2k+2, 2k+1, 2k+1)$ for $k = \frac{1}{2}(h-1) \in \mathbb{Z}^+$. \square

In what follows, let $\Pi = \Pi(n)$ denote an $n \times n$ permutation matrix corresponding to the n -cycle $(1, 2 \cdots n)$ given by

$$\Pi = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 1 \\ 1 & 0 & 0 & \dots & \dots & 0 \end{bmatrix}.$$

Then Π^2 is the permutation matrix of another n -cycle, but with the diagonal moved up to the second off-diagonal, and so on, until $\Pi^n = I$.

Lemma 4.10. *For a positive integer s , let L be the $(2s+2) \times (2s+2)$ -matrix equals to $\Pi + \Pi^2 + \dots + \Pi^s$ where $\Pi = \Pi(2s+2)$ as defined above. Then*

$$M(L) = \begin{bmatrix} L & L^T + I \\ L + I & L^T \end{bmatrix}$$

is an adjacency matrix of a DSRG with parameters $(4(s+1), 2s+1, s+1, s, s)$.

Proof. First we will show that $MJ = JM = kJ = (2s+1)J$

$$MJ = \begin{bmatrix} L\bar{J} + L^T\bar{J} + \bar{J} & L\bar{J} + L^T\bar{J} + \bar{J} \\ L\bar{J} + L^T\bar{J} + \bar{J} & L\bar{J} + L^T\bar{J} + \bar{J} \end{bmatrix} = JM$$

Since we know that the matrix L has s ones per row we can transform

$$MJ = JM = \begin{bmatrix} \bar{J}(s+s+1) & \bar{J}(s+s+1) \\ \bar{J}(s+s+1) & \bar{J}(s+s+1) \end{bmatrix} = (2s+1)J$$

Therefore the equation $MJ = JM = kJ$ holds.

The matrix L may be expressed as the block matrix

$$L = \begin{bmatrix} B & B^T \\ B^T & B \end{bmatrix}$$

with the $(s+1) \times (s+1)$ -matrix

$$B = \begin{bmatrix} 0 & 1 & \dots & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix}$$

the matrix with an all one upper triangle. It is easy to see that

$$L + L^T = \left[\begin{array}{c|c} B + B^T & B + B^T \\ \hline B + B^T & B + B^T \end{array} \right] = \left[\begin{array}{c|c} J - I & J - I \\ \hline J - I & J - I \end{array} \right];$$

$$L^2 = (L^T)^2 = \left[\begin{array}{c|c} B^2 + B^{T^2} & BB^T + B^TB \\ \hline BB^T + B^TB & B^2 + B^{T^2} \end{array} \right];$$

$$LL^T = L^T L = \begin{bmatrix} BB^T + B^T B & B^2 + B^{T^2} \\ B^2 + B^{T^2} & BB^T + B^T B \end{bmatrix};$$

and thus,

$$L^2 + LL^T + L + L^T = sJ_{2s+2}$$

As in Lemma 4.8,

$$M^2 = \left[\begin{array}{c|c} L^2 + L^T L + L + L^T + I & LL^T + L + L^T + L^{T^2} \\ \hline L^2 + L^T L + L + L^T & LL^T + L + L^T + L^{T^2} + I \end{array} \right].$$

Using the above simplification, we can transform M^2 into

$$M^2 = \left[\begin{array}{c|c} sJ_{2s+2} + I_{2s+2} & sJ_{2s+2} \\ \hline sJ_{2s+2} & sJ_{2s+2} + I_{2s+2} \end{array} \right] = sJ_{4s+4} + I_{4s+4}.$$

Therefore, M is the adjacency matrix of a DSRG with parameters $(4(s+1), 2s+1, s+1, s, s)$. \square

5 Isomorphisms between Constructions

We have discussed many different construction methods in searching for new DSRGs. We have seen that graphs with the same parameters can be obtained from different construction methods. In this section we investigate whether our construction methods are well-defined in the sense of isomorphic tournaments producing isomorphic graphs and whether the graphs having the same parameters are isomorphic.

Theorem 5.1. *All constructions of DSRGs discussed in Lemmas 4.2, 4.3, 4.4, 4.8, 4.9 and 4.10 are well defined constructions.*

Proof. Lemmas 4.2 and 4.3: If $A \cong B$; that is, if there exists a permutation matrix P such that $PAP^{-1} = B$, then

$$\begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} A & A^T \\ A & A^T \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix} = \begin{bmatrix} B & B^T \\ B & B^T \end{bmatrix}$$

so $\begin{bmatrix} A & A^T \\ A & A^T \end{bmatrix} \cong \begin{bmatrix} B & B^T \\ B & B^T \end{bmatrix}$, and similarly, for Lemma 4.2. The same permutation matrix works for Lemma 4.3. Since

$$\begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} A & A^T + I \\ A + I & A^T \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix} = \begin{bmatrix} B & B^T + I \\ B + I & B^T \end{bmatrix}$$

the constructions again produce isomorphic graphs. This also shows that for any matrices A, B where $A \cong B$, $M(A) \cong M(B)$.

Lemma 4.4: If $PAP^{-1} = B$, the obvious generalization of the smaller case,

$$\begin{bmatrix} P & 0 & 0 & \dots & 0 \\ 0 & P & 0 & \dots & 0 \\ 0 & 0 & P & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & P \end{bmatrix} \begin{bmatrix} A & A^T & A & \dots & A^T \\ A & A^T & A & \dots & A^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A & A^T & A & \dots & A^T \end{bmatrix} \begin{bmatrix} P^{-1} & 0 & 0 & \dots & 0 \\ 0 & P^{-1} & 0 & \dots & 0 \\ 0 & 0 & P^{-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & P^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} B & B^T & B & \dots & B^T \\ B & B^T & B & \dots & B^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & B^T & B & \dots & B^T \end{bmatrix}$$

shows that again we have isomorphic graphs.

Lemmas 4.8, 4.9 and 4.10: If we show that given $A \cong B$, $D(A) \cong D(B)$ and $M(A) \cong M(B)$, it follows that these constructions produce isomorphic graphs given isomorphic tournaments, or block matrices in the case of Lemma 4.10. Since the proof of Lemma 4.3 shows $M(A) \cong M(B)$, it only remains to show $D(A) \cong D(B)$. This is done by seeing that

$$\begin{bmatrix} 1 & & & & \\ & P & & & \\ & & 1 & & \\ & & & & P \end{bmatrix} \begin{bmatrix} 0 & \mathbf{1}^T & 0 & \mathbf{0}^T \\ \mathbf{0} & A & \mathbf{1} & A^T \\ 0 & \mathbf{0}^T & 0 & \mathbf{1}^T \\ \mathbf{1} & A^T & \mathbf{0} & A \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & P^{-1} & & & \\ & & 1 & & \\ & & & & P^{-1} \end{bmatrix} \\ = \begin{bmatrix} 0 & \mathbf{1}^T & 0 & \mathbf{0}^T \\ \mathbf{0} & B & \mathbf{1} & B^T \\ 0 & \mathbf{0}^T & 0 & \mathbf{1}^T \\ \mathbf{1} & B^T & \mathbf{0} & B \end{bmatrix}.$$

□

So, using the same construction, isomorphic tournaments always produce isomorphic DSRGs, but it is not as clear if, for one construction, non-isomorphic tournaments always yield non-isomorphic DSRGs.

With the two very similar constructions from Lemma 4.2, it is natural to wonder when, given the same tournament, these two different constructions actually produce isomorphic graphs. Below is one criterion for determining if they are isomorphic.

Lemma 5.2. *Let A be an adjacency matrix of a regular tournament and B, C be the adjacency matrices of the DSRGs constructed from Lemmas 4.2(1) and 4.2(2), respectively. If there exists a permutation matrix P such that $PA = A^T = AP$, then $B \cong C$.*

Proof. Assuming $PA = A^T = AP$, so $P^{-1}A^T = A = A^T P^{-1}$, if

$$\begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} A & A^T \\ A & A^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & P^{-1} \end{bmatrix} = \begin{bmatrix} A & A^T P^{-1} \\ PA & PAP^{-1} \end{bmatrix} = \begin{bmatrix} A & A \\ A^T & A^T \end{bmatrix} = C.$$

So $B \cong C$

□

Since circulant matrices commute with each other, this condition is reduced to $PA = A^T$, which happens iff each row of A appears also as a column of A . This property shows up in many tournaments that can be decomposed as a sum of permutation matrices representing h -cycles where $|A| = 2h + 1$.

Proposition 5.3. *Let $\Pi = \Pi(2k + 1)$. The matrices defined as $P = \Pi + \Pi^3 + \Pi^5 + \dots + \Pi^{2k-1}$, $P_0 = \Pi^2 + \Pi^4 + \Pi^6 + \dots + \Pi^{2k}$, and $P_j = \Pi^j(\Pi + \Pi^2 + \Pi^3 + \dots + \Pi^k)$ for any $0 \leq j \leq k$.*

Just as Lemma 4.4 generalizes our construction in Lemma 4.2, the sufficient condition for an isomorphism between the two DSRGs constructed from a regular tournament A in Lemma 5.2 holds in the more general case.

Lemma 5.4. *Let A be an adjacency matrix of a regular tournament and B, C be the adjacency matrices of the DSRGs constructed from Lemmas 4.8(1) and 4.8(2), respectively. If there exists a permutation matrix P such that $PA = A^T = AP$, then $B \cong C$.*

Proof. If

$$H = \begin{bmatrix} I & 0 & 0 & \dots & 0 \\ 0 & P & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & P \end{bmatrix}$$

where 0 is the matrix with all-0 entries, then $HBH^{-1} = C$, since each 2-block \times 2-block section reduces to the matrix from Lemma 5.2. \square

It is interesting to note that the block columns of B (block rows of C) could actually be arranged in any order and would still produce a DSRG isomorphic to the original graph constructed in Lemma 4.8. This isomorphism is associated with the permutation matrix $P \otimes I_{2h+1}$, for some permutation matrix P of order $2w$.

6 Areas of Further Investigation and Summary

We conclude the paper by making a few remarks.

1. Using Cayley graphs to construct DSRGs requires us to choose the correct subset S . Looking at the multiplication table for the set S determines if it works, but what is needed is a way to eliminate possible subsets before constructing their table. **Question:** Short of trying every possible subset, is there a simple way to pick an S that would generate a DSRG, or at least a better set of criteria so a much smaller set of subsets would need to be tested?
2. In using tournaments for all of these constructions, we became interested in tournaments themselves. Questions about how many regular tournaments exist came up, but this is still open and complete results only exist for small orders. Also, if a regular tournament is a circulant matrix, they become easier to work with, but not all tournaments are circulants. But, if any regular tournament is isomorphic to a circulant matrix, which means we can always choose to work with a circulant representative, our search can be simplified by allowing us to work matrices with several nice properties. A helpful paper on tournaments was [3].
3. Since we've shown that isomorphic tournaments, when put through the same construction, will also produce isomorphic graphs, but have found that, for small orders, non-isomorphic tournaments yield non-isomorphic graphs, we would like to consider how many different classes of non-isomorphic graphs are created in each case. **Conjecture: For any construction method C , where $C(A)$ is the graph resulting from using matrix A in C , $A \cong B$ iff $C(A) \cong C(B)$.** If this conjecture is true, each parameter set satisfied by our constructions has at least one non-isomorphic graph for each regular tournament, giving us good lower bounds for many parameter sets. There are a lot of non-isomorphic regular tournaments, especially as the order of the tournaments gets large. To see exactly how much this gets us, there are already 1223 non-isomorphic regular tournaments of order 11 and this increases greatly to 1495297 non-isomorphic tournaments of order 13. A complete list of these is found in [9]. If this conjecture is not true, what characteristics of the tournaments are involved in creating isomorphic graphs? Thinking about how this could happen, the tournaments would have to be similar enough to make isomorphic graphs, yet not be isomorphic themselves,

which seems improbable. However, with as many regular tournaments as there are, it is certainly possible.

4. Almost all of these various constructions are done with regular tournaments, but this may not be the weakest condition we could impose to ensure that our constructions still leave us with DSRGs. There could be a more inclusive class of matrices that also work in these constructions, allowing for even more non-isomorphic graphs to be created.
5. The reason that isomorphisms between the resulting graphs is important is that none of the parameter sets able to be produced by our constructions are new. They have all been covered by previous constructions, most notably Jørgensen's construction in [6]. So, instead of finding new parameter sets, we've provided several ways of realizing these parameter sets.
6. How does isomorphism of tournaments transfer to isomorphism of graphs through the various constructions? It doesn't necessarily carry from one construction to another. We have one sufficient condition that guarantees isomorphism, but this may not also be a necessary condition in this case. Our last 3 constructions are also very similar, often making DSRGs with the same parameters. The similarity of form in the final adjacency matrix, although not necessarily in construction of that matrix, may lead to a number of isomorphic graphs. These block matrix constructions use matrices of the same form,

$$M(B) = \begin{bmatrix} B & B^T + I \\ B + I & B^T \end{bmatrix},$$

but have different restrictions on what kind of matrix, B , makes up each block. There is possibly a more general way to use this form of matrix to construct DSRGs that includes all of our constructions. This relates back to the question of whether having our blocks based on regular tournaments is the weakest possible condition. The single form $M(B)$, when given various types of matrices, creates DSRGs. Finding more such forms of block matrices, or finding all possible matrices B so $M(B)$ is a DSRG are possible routes towards discovering more directed strongly regular graphs.

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7 Lists of Directed Strongly Regular Graphs

Table 1: Our Constructions of DSRGs

Lemma	Source	DSRG($v, k, \mu + 1, \mu, \mu$)	DSRG($v, k, \mu, \mu - 1, \mu$)
4.2	A regular tournament with adjacency matrix A and a DSRG with adjacency matrix $M(A) = \begin{bmatrix} A & A^T \\ A & A^T \end{bmatrix}$ or $M(A) = \begin{bmatrix} A & A \\ A^T & A^T \end{bmatrix}$		$(4k + 2, 2k, k, k - 1, k)$
4.3	A regular tournament with adjacency matrix A and a DSRG with adjacency matrix $M(A) = \begin{bmatrix} A & A^T + I \\ A + I & A^T \end{bmatrix}$		$(4k + 2, 2k + 1, k + 1, k, k + 1)$
4.4	A regular tournament with adjacency matrix A and a DSRG with adjacency matrix $M(A) = \begin{bmatrix} A & A^T & A & \dots & A^T \\ A & A^T & A & \dots & A^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A & A^T & A & \dots & A^T \end{bmatrix}$ or $M(A) = \begin{bmatrix} A & A & \dots & A \\ A^T & A^T & \dots & A^T \\ \vdots & \vdots & \ddots & \vdots \\ A^T & A^T & \dots & A^T \end{bmatrix}$		$(w(4k + 2), 2wk, wk, w(k - 1)wk)$
4.8	$D(T)$ the adjacency matrix of a doubly regular $(m, 2)$ team tournament and a DSRG with adjacency matrix $M(D) = \begin{bmatrix} D & D^T + I \\ D + I & D^T \end{bmatrix}$	$(4m, 2m - 1, m, m - 1, m - 1)$ where $m \equiv 0 \pmod{4}$	
4.9	A regular tournament with adjacency matrix A , DSRG with adjacency matrix $M(D) = \begin{bmatrix} D & D^T + I \\ D + I & D^T \end{bmatrix}$	$(4n + 4, 2n + 1, n + 1, n, n)$ where n is odd	
4.10	L is an $2s + 2 \times 2s + 2$ matrix equal to $\Pi + \Pi^2 + \dots + \Pi^s$ where Π^n is an $n \times n$ matrix as defined in Lemma 5.2, DSRG with adjacency matrix $M(L) = \begin{bmatrix} L & L^T + I \\ L + I & L^T \end{bmatrix}$	$(4(s + 1), 2s + 1, s + 1, s, s)$	

Table 2: DSRGs (v, k, t, λ, μ) Constructed by Lemma 4.2/Lemma 4.4 and DSRGs $(v', k', t', \lambda', \mu')$ Constructed from Lemma 4.3 with up to 34 Vertices

v	k	t	λ	μ	v'	k'	t'	λ'	μ'
6	2	1	0	1	6	3	2	1	2
10	4	2	1	2	10	5	3	2	3
14	6	3	2	3	14	7	4	3	4
18	8	4	3	4	18	9	5	4	5
22	10	5	4	5	22	11	6	5	6
26	12	6	5	6	26	13	7	6	7
30	14	7	6	7	30	15	8	7	8
34	16	8	7	8	34	17	9	8	9

Table 3: DSRGs Constructed from Lemmas 4.8, 4.9, and 4.10 with up to 32 Vertices

v	k	t	λ	μ	Consturction Method	Remark on Construction
8	3	2	1	1	Lemmas 4.9 and 4.10	This graph is unique and therefore Lemmas 4.9 and 4.10 Isomorphic
12	5	3	2	2	Lemma 4.10	
16	7	4	3	3	Lemmas 4.8, 4.9, and 4.10	Lemmas 4.8 and 4.9 come from one tournament so they are isomorphic, they are not isomorphic to Lemma 4.10
20	9	5	4	4	Lemma 4.10	
24	11	6	5	5	Lemmas 4.9 and 4.10	isomorphic (???)
28	13	7	6	6	Lemma 4.10	
32	15	8	7	7	Lemmas 4.8, 4.9, and 4.10	isomorphic (???)