Injection Structures and the Ershov Hierarchy

Francis Adams

Georgia State University

January 17, 2019

- *Equivalence Structures and Isomorphisms in the Difference Hierarchy* by Remmel, LaForte, and Cenzer
- *Computability-Theoretic Properties of Injection Structures* by Remmel, Harizanov, and Cenzer

Injection Structures

- Injection structures are (A, f) where f is an injection on A.
- A is partitioned into finite orbits, ω-orbits, and $\mathbb{Z}$-orbits.
- An injection structure $\mathcal{A}$ has a character, identifying its finite orbits.

$$\chi(\mathcal{A}) = \{(k, n) : n, k > 0 \text{ and } \mathcal{A} \text{ has at least } n \text{ orbits of size } k\}$$

- (A, f) is computable if both A and f are computable.

Computable Injection Structures: Existence

Say a set $K \subseteq \mathbb{N}^+ \times \mathbb{N}^+$ is a character if for all n and k, $(k, n+1) \in K$ implies $(k, n) \in K$.

Proposition (RHC)

- *If* (A, f) *is computable, then* $\chi(\mathcal{A})$ *is* Σ^0_1*.*
- *If* K *is a* Σ^0_1 *character, then there is a computable injection structure with character* K *and any number of* ω*-orbits or* $\mathbb{Z}$*-orbits.*

Computable Injection Structures: Uniqueness

Say a computable structure $\mathcal{A}$ is Δ^0_n-categorical if for any computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is a Δ^0_n isomorphism between them.

Theorem (RHC)

Let $\mathcal{A}$ be a computable injection structure.

- *$\mathcal{A}$ is computably categorical iff $\mathcal{A}$ only has finitely many infinite orbits.*
- *$\mathcal{A}$ is Δ^0_2 categorical iff $\mathcal{A}$ has only finitely many ω-orbits or finitely many $\mathbb{Z}$-orbits.*
- *$\mathcal{A}$ is Δ^0_3 categorical.*

Complexity of Injection Structures

Two ways to consider complexity of the structure:

- Let f be computable and restrict f to a set A where A is non-computable.

Complexity of Injection Structures

Two ways to consider complexity of the structure:

- Let f be computable and restrict f to a set A where A is non-computable.
- Let A be computable but allow f to be non-computable.

Ershov Hierarchy of Sets

A set A is Δ^0_2 if there is a computable function $g : \mathbb{N} \times \mathbb{N} \to \{0,1\}$ so $\chi_A(x) = \lim_{s\to\infty} g(x,s)$.

Within the Δ^0_2 sets, say A is $n - c.e.$ for $n > 1$ if there is g as above so $g(x,0) = 0$ and $\{s : g(x,s) \neq g(x,s+1)\}$ has size at most n. Also, say that A is $\omega - c.e.$ if there is a computable function g as above and a computable function b so $\{s : g(x,s) \neq g(x,s+1)\}$ has size at most $b(x)$.

This is also called the difference hierarchy, since A is $n - c.e.$ iff $A = Y \setminus Z$ where Y is $c.e.$ and Z is $(n-1) - c.e.$.

Ershov Hierarchy of Functions

A function f is Δ^0_2 if there is a computable function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ so $f(x) = \lim_{s\to\infty} g(x, s)$.

Within the Δ^0_2 functions, say f is $n - c.e.$ for $n > 1$ if there is g as above so $\{s : g(x, s) \neq g(x, s + 1)\}$ has size less than n.
Also, say that f is $\omega - c.e.$ if there is a computable function g as above and a computable function b so $\{s : g(x, s) \neq g(x, s + 1)\}$ has size at most $b(x)$.

Ershov Hierarchy of Functions

Say f is graph$-n-c.e.$ if the graph of f is $n-c.e.$, and similarly for graph$-\omega-c.e.$.

The notions of $\alpha-c.e.$ and graph$-\alpha-c.e.$ function were also studied independently by Khoussainov, Stephan, and Yang, including for α a computable ordinal greater than ω.

Ershov Hierarchy of Functions

Facts about $\alpha - c.e.$ functions from RLC:

- An $\alpha - c.e.$ function is graph$-\alpha - c.e.$.
- For all $n > 0$ there is an $(n+1) - c.e.$ function which isn't graph$-n - c.e.$.
- There is a graph$-2 - c.e.$ function which isn't $\omega - c.e.$.

Ershov Hierarchy of Functions

If we want to refine Δ_2^0-isomorphisms to isomorphisms within the Ershov hierarchy, graph$-\alpha - c.e.$ functions are more appropriate. This is because, from RLC, if f is graph$-\alpha - c.e.$ then so is f^{-1}, but there is a $2 - c.e.$ bijection $f : \mathbb{N} \to \mathbb{N}$ so f^{-1} isn't even $\omega - c.e.$.

A computable injection structure $\mathcal{A}$ is Δ_2^0 categorical iff $\mathcal{A}$ has only finitely many ω-orbits or finitely many $\mathbb{Z}$-orbits. Can we improve this?

Theorem (RHC)

There are two computable injection structures that are not $\omega - c.e.$ isomorphic.

Open Question: What about graph$-n-c.e.$ or $\alpha - c.e.$ for higher α?

Σ^0_1 Injection Structures

Proposition (RHC)

For any Σ^0_1 injection structure $\mathcal{A}$, there is a computable structure $\mathcal{B}$ and a computable isomorphism from $\mathcal{B}$ onto $\mathcal{A}$.

Π^0_1 Injection Structures

Theorem (RHC)

- *If $\mathcal{A}$ is a Π^0_1 injection structure, then $\chi(\mathcal{A})$ is Σ^0_2, and for any Σ^0_2 character K there is a Π^0_1 structure with character K and any number of infinite orbits.*
- *If $\mathcal{A}, \mathcal{B}$ are isomorphic Π^0_1 structures with only finitely many ω-orbits, then $\mathcal{A}$ and $\mathcal{B}$ are Δ^0_2 isomorphic.*

Π^0_1 Injection Structures

Proposition

If (A, f) and (B, g) are isomorphic Π^0_1 structures so f, g have no infinite orbits, then $\mathcal{A}, \mathcal{B}$ are graph$-2-c.e.$ isomorphic.

Let $A = \mathbb{N} \setminus C$ and $B = \mathbb{N} \setminus D$ where C, D are $c.e.$, enumerated in stages C_s, D_s. Define ϕ in stages ϕ_s:
At stage s:

- Find $O_f(i), O_g(j)$ for $i, j \leq s$.
- Enumerate C_s, D_s

Π_1^0 Injection Structures

Proposition

If (A, f) and (B, g) are isomorphic Π_1^0 structures so f, g have no infinite orbits, then $\mathcal{A}, \mathcal{B}$ are graph$-2-c.e.$ isomorphic.

- If any orbits $O_f(i), O_g(j)$ intersect C_s, D_s, any offending pairs in ϕ_s are removed and are unavailable.
- If some orbit $O_f(i)$ is unmapped, available to be used, and of the same size as an untargeted, available orbit $O_g(j)$, then map the first orbit to the second using ϕ_s.

$n - c.e.$ Injection Structures

Proposition

- *If (A, f) and (B, g) are isomorphic $2 - c.e.$ structures so f, g have no infinite orbits, then $\mathcal{A}, \mathcal{B}$ are graph$-2 - c.e.$ isomorphic.*
- *If (A, f) and (B, g) are isomorphic $n - c.e.$ structures so f, g have no infinite orbits, then $\mathcal{A}, \mathcal{B}$ are graph$-2n - c.e.$ isomorphic.*

Work to Do

- What if the *function* is $n - c.e.$ instead of the underlying set?
- Strategy for negative results: Start with a standard computable structure and construct a Π^0_1 structure by removing orbits, in a way to defeat potential isomorphisms.

Thank you.