Cardinal Invariants of Topologically Presented Graphs

Francis Adams

Georgia State University

March 16, 2018

Many of these results are found in *Cardinal Invariants of Closed Graphs*, joint with Jindrich Zapletal.

Topologically Presented Graphs

- A graph on X is a symmetric, irreflexive $G \subseteq X^2$.
- A topologically presented graph is a pair (G, X) where G is a graph whose vertex set X is a topological space.
- A graph G on a topological space X is Borel if it is a Borel subset of X^2.
- For a graph G on X, $A \subseteq X$ is an anticlique (independent, discrete) if for any $x, y \in A$, $(x, y) \notin G$.
- For a graph G on X, $C \subseteq X$ is a clique if for all distinct $x, y \in C$ we have xGy.

Special Subsets of Topological Spaces

Let X be a topological space. Then $A \subseteq X$ is

- *Weakly Separated* if for all $x \in A$ there is a basic open set O_x containing x such that, for any $x, y \in A$ either $x \notin O_y$ or $y \notin O_x$.
- *Left Separated* if there is a wellordering $\prec$ of A so for all $x \in A$ there is a basic open set O around x such that for any $y \in A$ with $y \prec x$, we have $y \notin O$.

Special Subsets of Topological Spaces

Let X be a topological space. Then $A \subseteq X$ is

- *Discrete* if for all $x \in A$ there is a basic open set O around x containing no other elements of A.
- *Closed Discrete* if for all $x \in X$ there is a basic open set O around x containing no elements of A, except potentially x.

Special Subsets of Topological Spaces

Let X be a topological space. Then $A \subseteq X$ is

- *Weakly Separated* if for all $x \in A$ there is a basic open set O_x containing x such that, for any $x, y \in A$ either $x \notin O_y$ or $y \notin O_x$.
- *Left Separated* if there is a wellordering $\prec$ of A so for all $x \in A$ there is a basic open set O around x such that for any $y \in A$ with $y \prec x$, we have $y \notin O$.
- *Discrete* if for all $x \in A$ there is a basic open set O around x containing no other elements of A.
- *Closed Discrete* if for all $x \in X$ there is a basic open set O around x containing no elements of A, except potentially x.

Special Subsets of Topological Spaces

Each of these properties has their own covering invariant:

$$ws(X) \leq ls(X) \leq dis(X) \leq dis^*(X)$$

If X is metrizable and κ is infinite then $ws(X) \leq \kappa \Leftrightarrow dis^*(X) \leq \kappa$.

If X is such that closed sets are G_δ and κ is infinite then $dis(X) \leq \kappa \Leftrightarrow dis^*(X) \leq \kappa$.

Special Subsets of Topological Spaces

Let X be a topological space.

- If X is compact Hausdorff with no isolated points, then $cov(meager) \leq ls(X)$. (Gerlits, Juhász, Szentmiklóssy 2005)
- If X is compact hereditarily normal with no isolated points, then $2^{\aleph_0} \leq dis(X)$. (Juhász, Van Mill 2007)
- If X has an order topology and $Y \subseteq X$ is a locally compact, Lindelöf subspace, then $dis(Y) = |Y|$. (Spadaro 2009)

Special Subsets of Topological Spaces

Let X be a topological space. Then $A \subseteq X$ is

- *Weakly Separated* if for all $x \in A$ there is a basic open set O_x containing x such that, for any $x, y \in A$ either $x \notin O_y$ or $y \notin O_x$.
- *Left Separated* if there is a wellordering $\prec$ of A so for all $x \in A$ there is a basic open set O around x such that for any $y \in A$ with $y \prec x$, we have $y \notin O$.
- *Discrete* if for all $x \in A$ there is a basic open set O around x containing no other elements of A.
- *Closed Discrete* if for all $x \in X$ there is a basic open set O around x containing no elements of A, except potentially x.

Loose Sets

Definition

Let (G, X) be a topologically presented graph.
Say $A \subseteq X$ is G-loose if for any $x \in X$ there is an open set O containing x so O contains no G-neighbors of x from A.

We could similarly define G-semi-loose, G-left separated, and G-weakly separated sets as analogues of discrete, left separated, and weakly separated sets.

Loose Sets

- Examples of G-loose sets are closed anticliques and closed (topologically) discrete sets.
- Let G on $\mathbb{R}^2$ with the Euclidean topology be xGy iff $d(x, y) = 1$. Then $\mathbb{R}^2$ is G-loose.

For a graph G on a space X, the G-loose sets form an ideal on X.

Loose Number of Topologically Presented Graphs

Definition

Let (G, X) be a topologically presented graph. The loose number of (G, X), denoted by $\lambda(G, X)$, is the least κ such that X can be covered by κ many G-loose sets.
If the topological space is understood, we will just say that the loose number of G is $\lambda(G)$.

Since the loose sets form an ideal, if $\lambda(G)$ is finite, then $\lambda(G) = 1$.

Invariants of Topologically Presented Graphs

For a topologically presented graph (X, G), we could similarly define the invariants $\lambda^*(X, G), ls(X, G), ws(X, G)$ as the least number of G-semi-loose, G-left separated, and G-weakly separated sets necessary to cover X. Then we get

$$ws(X, G) \leq ls(X, G) \leq \lambda^*(X, G) \leq \lambda(X, G)$$

Invariants of Topologically Presented Graphs

Proposition

Let X be metrizable and G a graph on X. TFAE:

- $\lambda(X, G) \leq \aleph_0$
- $\lambda^*(X, G) \leq \aleph_0$
- $ls(X, G) \leq \aleph_0$
- $ls(X, G) = 1$
- $ws(X, G) \leq \aleph_0$
- $ws(X, G) = 1$

Basic Properties of the Loose Number

Let X, Y be topological spaces.

i) If G is a graph on X, H is a graph on Y and $f : Y \to X$ is a continuous homomorphism, then $\lambda(H) \leq \lambda(G)$.

ii) If G, H are graphs on X, Y respectively, then $\lambda(G \times H, X \times Y) = \lambda(G) \cdot \lambda(H) = max\{\lambda(G), \lambda(H)\}$.

iii) If G is a graph on X and Y is a finer topological space than X, then $\lambda(G, Y) \leq \lambda(G, X)$.

Minimal Topological Requirements

Let X be a T_1 space and G a graph on X.

- For each $x \in X$, $\{x\}$ is G-loose. In particular, the loose number of G is defined and $\lambda(G) \leq |X|$.
- If G is locally finite, then $\lambda(G) = 1$.

In contrast, for any graph G on a space X, $\lambda^*(X, G)$ is always defined and bounded above by $|X|$.

Chromatic Number

We can relate the loose number of a graph to its chromatic number, the least number of anticliques necessary to cover its vertex set. Recall that for a space X, its weight $w(X)$ is the least size of a basis for the topology of X.

Proposition

For a graph G on a space X, $\chi(G) \leq \lambda(G) \cdot w(X)$.
In particular, if X is second countable and $\lambda(G)$ is infinite, then $\chi(G) \leq \lambda(G)$.

Example

An example shows how the chromatic number and loose number are different, even on a compact Polish space:
Define the graph F on 2^ω where xFy iff x has finitely many 1's, and y agrees with x up to its last one (or conversely). With the usual product topology on 2^ω, $\lambda(F) = \mathfrak{d}$, but $\chi(F) = \aleph_0$.

Coloring Number

Definition (Erdös, Hajnal)

For a graph G on X, $Col(G)$ is the least κ such that there is a wellordering $\prec$ of X such that for any $x \in X$, $|\{y \prec x : xGy\}| < \kappa$.

It always holds that $\chi(G) \leq Col(G)$.
A graph G on X has countable coloring number if there is a wellordering so $\{y \prec x : xGy\}$ is always finite.

Countable Coloring Number

Recalling a previous proposition:
Let G be a graph on a metrizable space X. Then $\lambda(G) \leq \aleph_0$ iff there is a wellordering $\prec$ of X so for all $x \in X$ there is a basic open set O around x such that for any $y \prec x$, if xGy, then $y \notin O$.

Corollary

If X is metrizable and G is a graph on X with countable coloring number, then $\lambda(G) \leq \aleph_0$.

For the graph F in a previous example, $Col(F) = \aleph_1$ and $\lambda(F) = \mathfrak{d}$.

Examples

Graphs with countable coloring number, hence countable loose number on a metrizable space, include:

- Locally countable graphs
- Acyclic graphs
- Graphs generated by finitely many functions $f_1, \ldots, f_n$: xGy iff $f_i(x) = y$ or $f_i(y) = x$ for some $i \leq n$.

Examples

Examples of graphs with countable loose number which may not have countable coloring number:

For a metrizable space X, define the graph $G = G(X, d, D)$ on X by xGy iff $d(x, y) \in D$ where d is a metric on X and D is a set of positive reals.
By extending results of Komjath and Schmerl, we showed that for $n \in \omega$, D countable, and e the Euclidean metric, the graph $G(\mathbb{R}^n, e, D)$ has countable loose number.

Definability of Looseness

The most important context for considering the concept of looseness is for definable graphs on Polish spaces. Given this, there are some questions about this concept from the perspective of descriptive set theory:

- What is the descriptive complexity of the concept of being G-loose?
- What is the descriptive complexity of the graphs with countable loose number?
- What happens if we try to cover a space where the loose sets are required to be definable?

Definability of Looseness

We start by noting that for analytic graphs G, a set being G-loose is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. This means that for X, Y Polish, G an analytic graph on X, and $A \subseteq Y \times X$ analytic, $A_L = \{y \in Y : A_y \text{ is loose }\}$ is $\mathbf{\Pi}^1_1$. This holds since for some fixed enumeration of a basis $\{O_n : n \in \omega\}$ for X,

$$y \in A_L \text{ iff } \forall x \text{ either } (x, y) \notin A$$
$$\text{or } \exists n \forall z\, (x \in O_n \text{ and either } z \notin O_n \text{ or } (x, z) \notin G)$$

Definability of Looseness

Essentially the same computation shows that the collection of closed graphs on ω^ω with loose number 1 is a coanalytic subset of the hyperspace $K((\omega^\omega)^2)$. This is in fact the optimal complexity.

Theorem

The collection of closed graphs on ω^ω with loose number 1 is a complete coanalytic subset of $K((\omega^\omega)^2)$.

Definability of Looseness

Theorem

The collection of closed graphs on ω^ω with loose number 1 is a complete coanalytic subset of $K((\omega^\omega)^2)$.

Instead of working with ω^ω, work with $X = \omega^{<\omega} \cup \omega^\omega$ with a topology on generated by sets $O_t = \{x \in X : t \subseteq x\}$ and singletons $\{t\}$ for $t \in \omega^{<\omega}$.
For a tree $T \subseteq \omega^{<\omega}$, define a graph G_T on X so $(s, v) \in G_T$ iff $s \in T$ and $s \subseteq v$ (or conversely). The map $T \to G_T$ is a continuous function from the space of trees to $K(X^2)$, reducing the set of well-founded trees on ω to the set of closed graphs with loose number 1.

Definable Looseness

For an analytic graph G, the collection of G-loose sets is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. A consequence is that any analytic G-loose set is contained in a Borel G-loose set. This suggests we consider the following definition.

Definition

For a graph G on a space X, define $\lambda_B(G)$ to be the least κ such that X can be covered by κ many Borel G-loose sets.

G_0

There is a graph G_0 on 2^ω which is closed, locally countable, acyclic, has uncountable Borel chromatic number (since every nonmeager Borel set has a G_0 edge), and the following theorem holds:

G_0

There is a graph G_0 on 2^ω which is closed, locally countable, acyclic, has uncountable Borel chromatic number (since every nonmeager Borel set has a G_0 edge), and the following theorem holds:

Theorem (Kechris-Solecki-Todorcevic)

For an analytic graph G, exactly one of the following holds:

- *G has countable Borel chromatic number.*
- *There is a continuous homomorphism of G_0 to G.*

Borel Loose Number

Proposition

A nonmeager $B \subseteq 2^\omega$ with the Baire Property isn't G_0-loose. In particular, G_0 has uncountable Borel loose number.

We also know that if G is a graph on a space X, H is a graph on Y, and $f : Y \to X$ is a continuous homomorphism, then $\lambda_B(H) \leq \lambda_B(G)$.

Corollary

If an analytic graph G has countable Borel loose number, it must also have countable Borel chromatic number.

Borel Loose Number

- The closed graph F isn't loose, but has Borel chromatic number $\aleph_0$.
- The complete bipartite graph with parts $\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}$ is a Borel graph which isn't loose and has Borel chromatic number 2.
- If $f_1, \ldots, f_n$ are Borel functions, $\lambda_B(G_{f_1,\ldots,f_n})$ is countable.

Application: Anticlique Ideals

Definition

Given a graph G on a Polish space X, define I_G to be the σ-ideal generated by the compact G-anticliques.

We want to compare the cardinal invariants $non(I_G), cov(I_G)$ of the σ-ideal to standard cardinal characteristics such as $\mathfrak{b}$.

Consistency Results

Theorem (Geschke)

Let G be a closed graph on a Polish space X. Then either G has a perfect clique or there is a ccc forcing extension where $\mathrm{cov}(I_G) = \aleph_1$ and $\mathfrak{c}$ is arbitrarily large.

Consistency Results

Theorem (Geschke)

Let G be a closed graph on a Polish space X. Then either G has a perfect clique or there is a ccc forcing extension where $\mathrm{cov}(I_G) = \aleph_1$ and $\mathfrak{c}$ is arbitrarily large.

Theorem (A., Zapletal)

Let G be a closed graph on a compact Polish space X. If G has countable loose number, then in some generic extension $\mathfrak{b} < \mathrm{non}(I_G)$.

Thank you.