Loose Number of Topologically Presented Graphs

Francis Adams

Georgia State University

March 3, 2018

Many of these results are found in *Cardinal Invariants of Closed Graphs*, joint with Jindrich Zapletal.

Topologically Presented Graphs

- A graph on X is a symmetric, irreflexive $G \subseteq X^2$.
- A topologically presented graph is (G, X) where G is a graph whose vertex set X is a topological space.
- A graph G on a topological space X is Borel if it is a Borel subset of X^2.
- For a graph G on X, $A \subseteq X$ is an anticlique (independent, discrete) if for any $x, y \in A$, $(x, y) \notin G$.
- For a graph G on X, $C \subseteq X$ is a clique if for all distinct $x, y \in C$ we have xGy.

Special Subsets of Topological Spaces

Let X be a topological space. Then $A \subseteq X$ is

- *Weakly Separated* if for all $x \in A$ there is a basic open set O_x containing x such that, for any $x, y \in A$ either $x \notin O_y$ or $y \notin O_x$.
- *Left Separated* if there is a wellordering $\prec$ of A so for all $x \in A$ there is a basic open set O around x such that for any $y \in A$ with $y \prec x$, we have $y \notin O$.

Special Subsets of Topological Spaces

Let X be a topological space. Then $A \subseteq X$ is

- *Discrete* if for all $x \in A$ there is a basic open set O around x containing no other elements of A.
- *Closed Discrete* if for all $x \in X$ there is a basic open set O around x containing no elements of A, except potentially x.

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Loose Sets

Definition

Let (G, X) be a topologically presented graph.
Say $A \subseteq X$ is G-loose if for any $x \in X$ there is an open set O containing x so O contains no G-neighbors of x from A.

- Examples of G-loose sets are closed anticliques and closed (topologically) discrete sets.
- Let G on $\mathbb{R}^2$ with the Euclidean topology be xGy iff $d(x, y) = 1$. Then $\mathbb{R}^2$ is G-loose.

For a graph G on a space X, the G-loose sets form an ideal on X.

Loose Number of Topologically Presented Graphs

Definition

Let (G, X) be a topologically presented graph. The loose number of (G, X), denoted by $\lambda(G, X)$, is the least κ such that X can be covered by κ many G-loose sets.
If the topological space is understood, we will just say that the loose number of G is $\lambda(G)$.

Since the loose sets form an ideal, if $\lambda(G)$ is finite, then $\lambda(G) = 1$.

Basic Properties

Let X, Y be topological spaces.

i) If G is a graph on X, H is a graph on Y and $f : Y \to X$ is a continuous homomorphism, then $\lambda(H) \leq \lambda(G)$.

ii) If G, H are graphs on X, Y respectively, then $\lambda(G \times H, X \times Y) \leq \lambda(G) \cdot \lambda(H)$.

iii) If G is a graph on X and Y is a finer topological space than X, then $\lambda(G, Y) \leq \lambda(G, X)$.

Minimal Topological Requirements

Let X be a T_1 space and G a graph on X.

- For each $x \in X$, $\{x\}$ is G-loose. In particular, the loose number of G is defined and $\lambda(G) \leq |X|$.
- If G is locally finite, then $\lambda(G) = 1$.

Chromatic Number

We can relate the loose number of a graph to its chromatic number, the least number of anticliques necessary to cover its vertex set.

Proposition

For a graph G on a space X, $\chi(G) \leq \lambda(G) \cdot w(X)$.
In particular, if X is second countable and $\lambda(G)$ is infinite, then $\chi(G) \leq \lambda(G)$.

Example

An example shows how the chromatic number and loose number are different, even on a compact Polish space:
Define the graph F on 2^ω where xFy iff x has finitely many 1's, and y agrees with x up to its last one (or conversely). With the usual product topology on 2^ω, $\lambda(F) = \mathfrak{d}$, but $\chi(F) = \aleph_0$.

Coloring Number

Definition (Erdös, Hajnal)

For a graph G on X, $Col(G)$ is the least κ such that there is a wellordering $\prec$ of X such that for any $x \in X$, $\{y \prec x : xGy\}$ has size less than κ.

It always holds that $\chi(G) \leq Col(G)$.
A graph G on X has countable coloring number if there is a wellordering so $\{y \prec x : xGy\}$ is always finite.

Countable Coloring Number

Theorem

Let G be a graph on a metrizable space X. Then the following are equivalent:

1. $\lambda(G) \leq \aleph_0$.
2. *For all $x \in X$ there is a basic open set O_x containing x such that, for any $x, y \in X$, if xGy, then either $x \notin O_y$ or $y \notin O_x$.*
3. *There is a wellordering $\prec$ of X so for all $x \in X$ there is a basic open set O around x such that for any $y \prec x$, if xGy, then $y \notin O$.*

So graphs on metrizable spaces with countable coloring number have countable loose number.

Examples

Graphs with countable coloring number, hence countable loose number on a metrizable space, include:

- Locally countable graphs
- Acyclic graphs
- Graphs generated by finitely many functions $f_1, \ldots, f_n$: xGy iff $f_i(x) = y$ or $f_i(y) = x$ for some $i \leq n$.

Examples

Examples of graphs with countable loose number which may not have countable coloring number:

For a metrizable space X, define the graph $G = G(X, d, D)$ on X by xGy iff $d(x, y) \in D$ where d is a metric on X and D is a set of positive reals.
By extending results of Komjath and Schmerl, we showed that for $n \in \omega$, D countable, and e the Euclidean metric, the graph $G(\mathbb{R}^n, e, D)$ has countable loose number.

Application: Anticlique Ideals

Definition

Given a graph G on a Polish space X, define I_G to be the σ-ideal generated by the compact G-anticliques.

We want to compare the cardinal invariants $non(I_G), cov(I_G)$ of the σ-ideal to standard cardinal characteristics such as $\mathfrak{b}$.

Consistency Results

Theorem (Geschke)

Let G be a closed graph on a Polish space X. Then either G has a perfect clique or there is a ccc forcing extension where $cov(I_G) = \aleph_1$ and $\mathfrak{c}$ is arbitrarily large.

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Theorem (A., Zapletal)

Let G be a closed graph on a compact Polish space X. If G has countable loose number, then in some generic extension $\mathfrak{b} < non(I_G)$.

It is open whether these can always be separated at $\aleph_1$ and $\aleph_2$.

Thank you.