Computable Ultrahomogeneous Structures

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Computable Model Theory

All structures will be countable structures over a language with finitely many non-logical symbols.

Definition

A structure $\mathcal{A}$ is computable if its universe, functions, and relations are computable.

This is equivalent to saying the atomic diagram of $\mathcal{A}$ is computable.

Effective Categoricity

- A computable structure $\mathcal{A}$ is computably categorical if for every computable structure $\mathcal{B}$ which is isomorphic to $\mathcal{A}$, there is an isomorphism $\mathcal{A} \to \mathcal{B}$ which is computable.
- A computable structure $\mathcal{A}$ is relatively computably categorical if for every structure $\mathcal{B}$ which is isomorphic to $\mathcal{A}$, there is an isomorphism $\mathcal{A} \to \mathcal{B}$ which is computable relative to $\mathcal{B}$.

We can similarly define when a structure is (relatively) Δ^0_2 categorical.

Ultrahomogeneous Structures

Definition

A structure $\mathcal{A}$ is ultrahomogeneous if every partial isomorphism of finitely generated substructures $\psi : \langle \vec{x} \rangle \to \langle \vec{y} \rangle$ extends to an automorphism of $\mathcal{A}$.

Examples

- The canonical example of an ultrahomogeneous structure is the linear order $(\mathbb{Q}, <)$.
- An ultrahomogeneous equivalence structure (a set equipped with an equivalence relation) has all equivalence classes the same size.
- Other examples include the random graph and the countable atomless boolean algebra.

Weakly Ultrahomogeneous Structures

Definition

A structure $\mathcal{A}$ *is weakly ultrahomogeneous if it becomes ultrahomogeneous after adding constants to the language which name finitely many points. Call this finite set of named points an exceptional set of* $\mathcal{A}$*.*

Examples

The linear order consisting of a copy of $\mathbb{Q}$, followed by a finite linear order of length 5, followed by another copy of $\mathbb{Q}$ is weakly ultrahomogeneous. The 5 elements in the middle form an exceptional set.

$$\mathbb{Q} \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad \mathbb{Q}$$

Theorem

Every computable weakly ultrahomogeneous structure is relatively Δ^0_2-categorical.

Proof.

Use a back-and-forth argument. At each stage you will have to check if two finitely generated substructures are isomorphic, which is Π^0_1. □

Corollary

Any locally finite computable weakly ultrahomogeneous structure is relatively computably categorical. In particular this holds for relational structures.

Weakly Ultrahomogeneous Structures

For various classes of structures:

- Classify the weakly ultrahomogeneous structures.
- Compare weak ultrahomogeneity with the notions of effective categoricity.

Linear Orders

Theorem

A countable linear order $\mathcal{A}$ is weakly ultrahomogeneous iff $\mathcal{A}$ has finitely many successivities.

Corollary (Remmel 1981)

If $\mathcal{A}$ is a computable linear order, then $\mathcal{A}$ is weakly ultrahomogeneous iff $\mathcal{A}$ is computably categorical iff $\mathcal{A}$ is relatively computably categorical.

Equivalence Structures

Theorem

An equivalence structure $\mathcal{A}$ is weakly ultrahomogeneous iff all but finitely many equivalence classes are the same size. In this case, a minimal exceptional set contains exactly one element from each of the exceptional equivalence classes.

Corollary (Cenzer et al. 2005)

A computable equivalence structure is weakly ultrahomogeneous iff it is relatively computably categorical iff it is computably categorical.

Injection Structures

An injection structure is $\mathcal{A} = (A, f)$ where f is an injective function on A. Injection structures are partitioned into orbits of three types: finite cycles, ω-orbits, and $\mathbb{Z}$-orbits.

An injection structure is ultrahomogeneous iff it has no ω-orbits.

Injection Structures

Theorem

An injection structure is weakly ultrahomogeneous iff it has finitely many ω-orbits. In this case, a minimal exceptional set contains exactly one member from each ω-orbit.

Injection Structures

Theorem (Cenzer et. al. 2014)

- *A computable injection structure is computably categorical iff it is relatively computably categorical iff it has finitely many infinite orbits*
- *Such a structure is (relatively) Δ^0_2-categorical iff it has finitely many ω-orbits or finitely many $\mathbb{Z}$-orbits.*

So for computable injection structures, computable categoricity $\Rightarrow$ weak ultrahomogeneity $\Rightarrow$ Δ^0_2-categoricity. Neither implication can be reversed.

Trees

Consider rooted trees, viewed as subsets of $\omega^{<\omega}$ closed under initial segments with the empty string λ as a root.
These can be realized as:

- A partial order $(T, <)$ where $<$ well-orders the predecessors of each node.
- (T, f) where f is a predecessor function: $f(t \frown n) = t$ and $f(\lambda) = \lambda$.

In both formulations we assume the root is named.

Trees

For $x \in T$, let $T[x]$ be the tree of extensions of x in T.

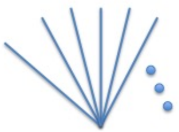
Let $T_n = \{x \in T : ht(x) = n\}$.

We define a rank on elements of T. For $x \in T$:

- $rk_T(x) = 0$ if x is a dead-end (leaf node)
- $rk_T(x) = \sup\{rk_T(y) + 1 : y \in T[x]\}$

Trees as Partial Orders

A tree $(T, <)$ is ultrahomogeneous iff T has rank ≤ 1.



Trees as Partial Orders

Proposition

$(T, <)$ is weakly ultrahomogeneous iff only finitely many elements have rank ≥ 1. In particular, weakly ultrahomogeneous trees have finite height.

Effective Categoricity of Trees

Theorem (Miller)

No tree $(T, <)$ *of infinite height is computably categorical.*

Theorem (Lempp et al)

A computable tree $(T, <)$ *is computably categorical iff it is relatively computably categorical iff it is of finite type.*

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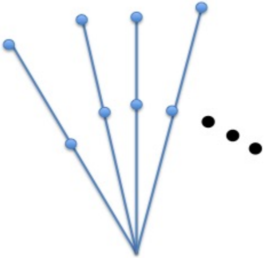

Trees with Predecessor

With a predecessor function we are able to determine the height of any node, so have more (weakly) ultrahomogeneous trees.

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(T, f) is ultrahomogeneous if for all n, all $a, b \in T_n$ have the same number of successors.

Equivalently there is a branching function $f : \omega \to \omega + 1$ such that every $a \in T_n$ has $f(n)$ many successors.

Trees with Predecessor

Given a tree T, a subtree $S \subseteq T$, and $x \in T$ we can define the subtree $T_S[x]$ consisting of the node x, its predecessors, together with all nodes in T extending successors of x which are not in S.

Proposition

(T, f) is weakly ultrahomogeneous iff there is a a finite subtree S of T such that, for every $x \in S$, the tree $T_S[x]$ is ultrahomogeneous.

Trees with Predecessor

Proposition

A tree (T, f) of height ≤ 2 is weakly ultrahomogeneous if and only if all but finitely many nodes of height 1 have an equal number of successors.

Proposition

A tree (T, f) of height 3 is weakly ultrahomogeneous if and only if the following conditions hold:

(a) *for each node x of height 1, all but finitely many successors of x have an equal number of successors;*

(b) *there are fixed h and k in $\omega \cup \{\omega\}$ such that all but finitely many nodes of height 1 have exactly h successors and each of those successors has exactly k successors.*

n-Equivalence Structures

Definition

For $n < \omega$, a nested n-equivalence structure is a structure $\mathcal{A} = (A, E_1, \ldots, E_n)$ where each E_i is an equivalence relation on A and for $i < j \leq n$ we have $xE_jy \rightarrow xE_iy$, i.e. $E_j \subseteq E_i$ as subsets of $A \times A$. Thus the E_i classes are partitioned by E_j.

n-Equivalence Structures

Definition

For any nested n-equivalence structure $\mathcal{A} = (A, E_1, \ldots, E_n)$, *let* $E_0 = A \times A$, *let* E_{n+1} *be equality, and define the tree* $T_{\mathcal{A}}$ *as follows. The universe of* $T_{\mathcal{A}}$ *is the set* $\{[a]_i : a \in A, i = 1, \ldots, n\}$ *and the partial ordering is inclusion. This means that for each* a *and* $i \leq n$, $[a]_i$ *is the predecessor of* $[a]_{i+1}$.

n-Equivalence Structures

Theorem (Marshall 2015)

Let $\mathcal{A}$ be a computable n-equivalence structure and $T_{\mathcal{A}}$ its corresponding tree of finite height. Then the following are equivalent:

- *$\mathcal{A}$ is computably categorical.*
- *$\mathcal{A}$ is relatively computably categorical.*
- *$(T_{\mathcal{A}}, \prec)$ is computably categorical.*
- *$(T_{\mathcal{A}}, \prec)$ is relatively computably categorical.*
- *$(T_{\mathcal{A}}, \prec)$ is of finite type.*

n-Equivalence Structures

Theorem

Let $\mathcal{A} = (A, E_1, \ldots, E_n)$ be a nested n-equivalence structure and let $E_0 = A \times A$ and E_{n+1} be equality. Then the following are equivalent.

1. *$\mathcal{A}$ is ultrahomogeneous.*
2. *For each $i \leq n$ there exists k_i such that every E_i class is partitioned into k_i many E_{i+1} classes.*
3. *$T_{\mathcal{A}}$ is ultrahomogeneous in the predecessor representation.*

n-Equivalence Structures

Corollary

If $\mathcal{A} = (A, E_1, \ldots, E_n)$ is a nested n-equivalence structure such that all equivalence classes are finite, then $\mathcal{A}$ is ultrahomogeneous if and only if each (A, E_i) is ultrahomogeneous.

This restriction is necessary as seen by the 2-equivalence structure where E_1 is two infinite classes and E_2 partitions one E_1 class into 3 classes and the other into 5 classes.

n-Equivalence Structures

Theorem

Let $\mathcal{A} = (A, E_1, \ldots, E_n)$ be a nested n-equivalence structure and let $E_0 = A \times A$ and E_{n+1} be equality. Then $T_{\mathcal{A}}$ is weakly ultrahomogeneous in the predecessor representation if and only if $\mathcal{A}$ is weakly ultrahomogeneous.

To come:

- Boolean algebras
- Abelian p-groups
- Permutations (two linear orders)
- Transfinite predecessor trees
- and more...

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Thank you.