Combinatorics of Borel Graphs on Polish Spaces

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Graphs on Polish Spaces

- A graph on X is a symmetric, irreflexive $G \subseteq X^2$.
- A graph G on a Polish space X is Borel if it is a Borel subset of X^2.
- $A \subseteq X$ is an anticlique (independent) if for any $x, y \in A$, $(x, y) \notin G$.
- $C \subseteq X$ is a clique if for all distinct $x, y \in C$ we have xGy.

Countably Chromatic Graphs

A graph G on a set X has countable chromatic number if $X = \bigcup A_n$ where each A_n is a G-anticlique.

Proposition

Let G be a graph on a second countable space X. TFAE:

- *G has countable chromatic number.*
- *$X = \bigcup A_n$ where for each $n \in \omega$ and each $x \in A_n$, there is an open set O containing x such that $O \cap A_n$ contains no G-neighbors of x.*

We get a stronger property by replacing the '$x \in A_n$' by '$x \in X$'.

σ-Loose Graphs

Definition

Let G be a graph on a space X.
Say $A \subseteq X$ is G-loose if for any $x \in X$ there is an open set O containing x so $O \cap A$ contains no G-neighbors of x.
Say G is σ-loose if $X = \bigcup A_n$ where each A_n is G-loose.

So a set A fails to be G-loose if there is a sequence $\{x_n : n \in \omega\} \subseteq A$ such that $x_n \to x$ and $x_n G x$ for all n.

Anticlique Ideals

Definition

Given a graph G on a Polish space X, define I_G to be the σ-ideal generated by the compact G-anticliques.

We want to compare the cardinal invariants $non(I_G), cov(I_G)$ of the σ-ideal to standard cardinal characteristics such as $\mathfrak{b}$ and $\mathfrak{d}$.

Consistency Results

Theorem (Geschke)

Let G be a closed graph on a Polish space X. Then either G has a perfect clique or there is a ccc forcing extension where $\mathrm{cov}(I_G) = \aleph_1$ and $\mathfrak{c}$ is arbitrarily large.

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Theorem

Let G be a closed graph on a K_σ Polish space X which is σ-loose in every generic extension. Then it is consistent that
$\aleph_1 = \mathfrak{b} < non(I_G) = \aleph_2 = \mathfrak{c}$.

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Say G is σ-loose if $X = \bigcup A_n$ where each A_n is G-loose.

Closure Properties

- If G_1, G_2 are σ-loose graphs on X, Y respectively, then $G_1 \times G_2$ is σ-loose on $X \times Y$.
- If G_1, G_2 are σ-loose graphs on X, then $G_1 \cup G_2$ is σ-loose on X.
- If H on Y is σ-loose, G is a graph on X and $f : X \to Y$ is a continuous homomorphism of G to H, then G is σ-loose.

Countable Coloring Number

Definition (Erdös, Hajnal)

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Proposition

For a graph G on a metric space X, G is σ-loose iff there is a wellordering $\prec$ of X such that for any $x \in X$, there is an open set O_x containing x so $\{y \prec x : xGy\} \cap O_x = \emptyset$.

So graphs with countable coloring number are σ-loose.

Consistency Results

We can remove the hypothesis 'σ-loose in every extension' in some cases.

Theorem (Zapletal)

There are Borel graphs K_0, K_1, K_2, K_3 with uncountable coloring number such that, for an analytic graph G on Polish X, exactly one of the following holds:

- *G has countable coloring number.*
- *There is a continuous injective homomorphism from one of the K_i into G.*

Consistency Results

Corollary

Let G be a closed graph on a compact Polish space X with countable coloring number. Then it is consistent that
$\aleph_1 = \mathfrak{b} < non(I_G) = \aleph_2 = \mathfrak{c}$.

Consistency Results

Corollary

Let G be a closed graph on a compact Polish space X with countable coloring number. Then it is consistent that
$\aleph_1 = \mathfrak{b} < non(I_G) = \aleph_2 = \mathfrak{c}$.

The graph G_0 introduced by Kechris – Solecki – Todorcevic which is the minimal analytic graph with uncountable Borel chromatic number is a closed, locally countable graph on 2^ω.

Nonexamples

If G has a perfect clique, then G is not σ-loose, $non(I_G) = \aleph_1$ and $cov(I_G) = \mathfrak{c}$.

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Example

Define the graph F on 2^ω by xFy if either x or y is constantly 0 after the point where they first differ. This is a closed graph with countable Borel chromatic number. However F is not σ-loose. Also, $non(I_F) \leq \mathfrak{b}$ and $cov(I_F) \geq \mathfrak{d}$.

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Example

E_0 on 2^ω is an F_σ countable equivalence relation, so E_0 is σ-loose. Also, $non(I_{E_0}) \leq \mathfrak{b}$ and $cov(I_{E_0}) \geq \mathfrak{d}$.

Metric Graphs

Let (X, d) be a Polish metric space, and let $D \subseteq (0, \infty)$. Define the graph $G = G(X, d, D)$ by xGy iff $d(x, y) \in D$.

Komjath and Schmerl show that for any countable D, and the Euclidean metric e on $\mathbb{R}^n$, the graph $G(\mathbb{R}^n, e, D)$ has countable chromatic number.

A central lemma from their proof can also be used to show

Proposition

For $n \in \omega$ and D countable, the graph $G(\mathbb{R}^n, e, D)$ is σ-loose.

Metric Graphs

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Proposition

Let (X, d) be a Polish ultrametric space, so $d(x, y) \leq max\{d(x, z), d(z, y)\}$, and let D be an 'appropriate' set of distances having 0 as a limit point. Then $G(X, d, D)$ has a perfect clique.

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Proposition

Let $X = [0, 1]^\omega$, let d be any compatible metric, and let D have 0 as a limit point. Then $G(X, d, D)$ is not σ-loose.

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- If G is a closed, non-σ-loose graph, must $non(I_G) \leq \mathfrak{b}$?
- Can we identify a basis for the family of non-σ-loose graphs under continuous homomorphisms? What about just the closed or analytic ones?

Thank you.